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THE t-REPRESENTATION OF THE GENERAL DEBYE FUNCTION

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Synopsis

The t-representation of the system function $S(p) = [1 + (p\tau_0)^{1-\alpha}]^{-\beta}$, *i.e.* the general Debye function, has been determined in the general case of $0 \leqq \alpha, \beta \leqq 1$ and $0 < \tau_0$ as an infinite (C, 1) summable series of Riesz distributions,

$$B_t = \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta + n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}.$$

It is established that a transformation T characterising a physical system and mapping an excitation f into a response r by $r = T[f] = B * f$ in spite of the singularities of the system function $S(p)$ possesses the six properties of (*i*) single valuedness, (*ii*) linearity, (*iii*) stationaryness, (*iv*) continuity, (*v*) passivity, and (*vi*) causality.

The main results required from the theory of distributions are presented in the text.

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1. Introduction

In the present paper the transformation of an excitation f into a response r, $T: f \rightarrow r$, as effected by a physical system, is considered from a general, mathematical point of view. Only linear systems are treated.

Often, rather than describing the transformation directly, *i.e.* by giving the t-representation, an integral transformation (Laplace transformation) may be carried out, whereby the equivalent p-representation is obtained.

Whereas in the t-representation the information about the transformation is inherent in the mathematical properties of the transformation, in the p-representation this information is contained in the properties of the system function $S(p)$, a complex valued function of a complex variable p.

As particular instances of a system function $S(p)$ are discussed the Debye function, $\dfrac{1}{1 + p\tau_0}$, and the general Debye function, $\dfrac{1}{[1 + (p\tau_0)^{1-\alpha}]^{\beta}}$, where $p \in \boldsymbol{C}^1$, $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1$, $0 \leqq \alpha, \beta \leqq 1$, and $0 < \tau_0$. Inspection shows that whereas the Debye function in the complex p-plane as its sole singularity has one simple pole, *viz.* $p = \dfrac{-1}{\tau_0}$, the general Debye function may possess branch point singularities which may be greater than one in number and which may be dense on the circle $|p| = \dfrac{1}{\tau_0}$. Furthermore, the general Debye function may have a branch point at $p = 0$.

Rather than as functions the excitations f and the responses r are treated as distributions, *i.e.* as elements of a topological, linear vector space $\mathscr{D}'$ (over $\boldsymbol{C}^1$) of continuous, linear mappings from a space of test functions into the field $\boldsymbol{C}^1$ of the complex numbers.

In section 2 the main results required from the theory of distributions[1])

[1]) For a full treatment of the theory of distributions see ref.s (5), (14), (15), (16), and also ref.s (2), (6), and (13).

are indicated. In sections 3 and 4 the relationships between the properties of a transformation T and the properties of the corresponding system function $S(p)$ are studied. Section 5 treats the properties of the general Debye function as system function and the properties of the transformation T corresponding to it. In section 6 the t-representation of the general Debye function as system function is determined.

The main results of the treatise are the theorems 5.1 and 6.1.

2. Distributions

Distributions[2)] are essentially defined as elements of topological vector spaces Φ' which are dual to certain other topological vector spaces Φ, the elements of which are termed test functions φ. The vector spaces Φ and Φ' are defined over the field $\boldsymbol{C}^1$ of the complex numbers. The test functions φ are defined pointwise on sets of points, *e.g.* $\boldsymbol{R}^1$, $\boldsymbol{C}^n$, but only complex valued functions defined on $\boldsymbol{R}^1$ will be required.

In what follows some important function spaces and their duals will be treated.

Let the vector space $\mathscr{C}(K)$ be the space of all continuous, complex valued functions, φ_K: $\boldsymbol{R}^1 \to \boldsymbol{C}^1$, of one real variable, which have their supports contained in the one compact set K, $K \subset \boldsymbol{R}^1$, $\operatorname{supp}\varphi_K \subseteqq K$. The vector space $\mathscr{C}$ of all continuous functions φ with compact support is generated as the union of all the vector subspaces $\mathscr{C}(K)$ as the compact set K varies over $\boldsymbol{R}^1$. By defining the seminorms $p(\varphi_K) = \sup\limits_{x \in K} |\varphi_K(x)|$ in the subspaces $\mathscr{C}(K)$ a topology of compact convergence is introduced on the spaces $\mathscr{C}(K)$. A topology on $\mathscr{C}$ is now defined as a set of neighbourhoods in $\mathscr{C}$ such that for each K the intersection of each neighbourhood and the space $\mathscr{C}(K)$ is a neighbourhood in $\mathscr{C}(K)$ under the above topology of compact convergence. As K varies over $\boldsymbol{R}^1$ this topology becomes the inductive limit topology of compact convergence, and the space $\mathscr{C}$ with this topology is the inductive limit of the spaces $\mathscr{C}(K)$.

Definition 2.1 *$\mathscr{C} = \mathscr{C}_x$ is the topological vector space over $\boldsymbol{C}^1$ of all continuous, complex valued functions, φ: $\boldsymbol{R}^1 \to \boldsymbol{C}^1$, of one real variable, $x \in \boldsymbol{R}^1$, which have compact support. The topology on $\mathscr{C}$ is the inductive limit topology of compact convergence.*

[2)] For a full treatment of the theory of distributions see ref.s (5), (14), (15), (16), and also ref.s (2), (6), and (13).

Theorem 2.1 *A sequence $\{\varphi_\nu\}$ of functions in $\mathscr{C}$, $\varphi_\nu \in \mathscr{C}$, $\nu = 1,2,3,\ldots$, converges to a limit φ in $\mathscr{C}$, $\varphi \in \mathscr{C}$, if and only if (i) there exists a space $\mathscr{C}(K)$ such that $\varphi, \varphi_\nu \in \mathscr{C}(K)$, $\nu = 1,2,3,\ldots$, and (ii) the sequence $\{\varphi_\nu\}$ converges to φ in the topology on $\mathscr{C}(K)$.*

Likewise, let the vector space $\mathscr{D}(K)$ be the space of all infinitely differentiable, complex valued functions, $\varphi_K\colon \boldsymbol{R}^1 \to \boldsymbol{C}^1$, of one real variable, $x \in \boldsymbol{R}^1$, which have their supports contained in the compact set K, $K \subset \boldsymbol{R}^1$, $\operatorname{supp}\varphi_K \subseteqq K$. The vector space $\mathscr{D}$ of all infinitely differentiable functions φ with compact support is generated as the union of all the vector subspaces $\mathscr{D}(K)$ as the compact set K varies over $\boldsymbol{R}^1$. By proceeding in a manner quite analogous to the above a topology of compact convergence is defined on the vector subspaces $\mathscr{D}(K)$ by defining the seminorms $p_j(\varphi_K) = \sup_{x\in K} |\varphi_K^{(j)}(x)|$, $j = 0,1,2,3,\ldots$, in the spaces $\mathscr{D}(K)$. Again it is possible to define in $\mathscr{D}$ a set of neighbourhoods such that for each K the intersection of each neighbourhood in $\mathscr{D}$ with the space $\mathscr{D}(K)$ is a neighbourhood in $\mathscr{D}(K)$ under the above topology of compact convergence, and the space $\mathscr{D}$ is thus generated as the inductive limit of the spaces $\mathscr{D}(K)$.

Definition 2.2 *$\mathscr{D} = \mathscr{D}_x$ is the topological vector space over $\boldsymbol{C}^1$ of all infinitely differentiable, complex valued functions, $\varphi : \boldsymbol{R}^1 \to \boldsymbol{C}^1$, of one real variable, $x \in \boldsymbol{R}^1$, which have compact support. The topology on $\mathscr{D}$ is the inductive limit topology of compact convergence.*

Theorem 2.2 *A sequence $\{\varphi_\nu\}$ of functions in $\mathscr{D}$, $\varphi_\nu \in \mathscr{D}$, $\nu = 1,2,3,\ldots$, converges to a limit φ in $\mathscr{D}$, $\varphi \in \mathscr{D}$, if and only if (i) there exists a space $\mathscr{D}(K)$ such that $\varphi, \varphi_\nu \in \mathscr{D}(K)$, $\nu = 1,2,3,\ldots$, and (ii) the sequence $\{\varphi_\nu\}$ converges to φ in the topology on $\mathscr{D}(K)$.*

Finally is introduced the space $\mathscr{S}$ of test functions of rapid decrease.

Definition 2.3 *$\mathscr{S} = \mathscr{S}_x$ is the topological vector space over $\boldsymbol{C}^1$ of all infinitely differentiable, complex valued functions, $\varphi\colon\boldsymbol{R}^1 \to \boldsymbol{C}^1$, of one real variable, $x \in \boldsymbol{R}^1$, with the property that $|x|^k \cdot |\varphi^{(l)}(x)| \leqq C_{kl}$, $k,l = 0,1,2,\ldots$, with C_{kl} a positive, real constant. The topology on $\mathscr{S}$ is introduced by defining the seminorms $p_{mn}(\varphi) = \sup_{x\in \boldsymbol{R}^1} |x^m \varphi^{(n)}(x)|$, $m,n = 0,1,2,3,\ldots$*

If Φ is a topological vector space over the field $\boldsymbol{C}^1$ of the complex numbers, the set of all continuous, linear mappings φ' of Φ into $\boldsymbol{C}^1$, $\varphi' : \Phi \to \boldsymbol{C}^1$, constitutes a vector space called the dual of Φ and denoted by Φ'. The value of the mapping $\varphi' \in \Phi'$ at the point $\varphi \in \Phi$ is denoted by $\varphi'(\varphi) = \langle \varphi', \varphi \rangle \in \boldsymbol{C}^1$.

To introduce a topology on Φ' let a topology be defined on Φ, and let $\mathscr{A}$ be a family of bounded subsets of Φ with the properties

(*i*) if $A \in \mathscr{A}$ and $B \in \mathscr{A}$ then there exists a $C \in \mathscr{A}$ such that $A \cup B \subseteqq C$,

(*ii*) if $A \in \mathscr{A}$ and λ is a complex number then there exists a $B \in \mathscr{A}$ such that $\lambda A \subseteqq B$.

A topology on Φ' now is determined by defining a set of seminorms in Φ' by $p'(\varphi') = \sup\limits_{\varphi \in A} |\langle \varphi', \varphi \rangle|$, $A \in \mathscr{A}$. In particular, two topologies on Φ' are of importance. If $\mathscr{A}$ is the set $\mathscr{A}_f$ of all finite subsets of Φ, the topology defined on Φ' is called the weak dual topology $\sigma(\Phi', \Phi)$. If $\mathscr{A}$ is the set $\mathscr{A}_b$ of all bounded subsets of Φ, the topology defined on Φ' is called the strong dual topology $\beta(\Phi', \Phi)$. As $\mathscr{A}_f \subseteqq \mathscr{A}_b$ it follows that $\sigma \subseteqq \beta$, *i.e.* that the strong topology is essentially finer than the weak topology.

Definition 2.4 *The dual space $\mathscr{C}'$ of the space $\mathscr{C}$ is the topological vector space of all continuous, linear, complex valued functionals, $\varphi' : \mathscr{C} \to \boldsymbol{C}^1$, defined on $\mathscr{C}$. This is the space of Radon measures on $\mathscr{C}$. The topologies on $\mathscr{C}'$ are the weak dual topology σ and the strong dual topology β.*

An important element in the space $\mathscr{C}'$ is the Dirac measure, the delta functional.

Definition 2.5 *The continuous, linear mapping $\delta : \varphi \to \varphi(x_0)$ of $\mathscr{C}$ into $\boldsymbol{C}^1$ is the delta functional at the point $x_0 \in \boldsymbol{R}^1$.*

The weak topology σ on the dual space Φ' is the topology of pointwise convergence in Φ. A sequence of functionals $\{\varphi'_\nu\}$, $\nu = 1, 2, 3, \ldots$, converges in the σ-topology to the limit φ' if and only if the sequence of complex numbers $\{\langle \varphi'_\nu, \varphi \rangle\}$ converges to the complex number $\langle \varphi', \varphi \rangle$ for every $\varphi \in \Phi$.

The strong topology β on the dual space Φ' is the topology of uniform convergence on every bounded subset of the space Φ. A sequence of functionals $\{\varphi'_\nu\}$, $\nu = 1, 2, 3, \ldots$, converges in the β-topology to the limit φ' if and only if the sequence of complex numbers $\{\langle \varphi'_\nu, \varphi \rangle\}$ converges to the complex number $\langle \varphi', \varphi \rangle$ uniformly on every bounded subset of Φ.

If a sequence of functionals converges in the strong β-topology then it also converges in the weak σ-topology.

Definition 2.6 *The dual space $\mathscr{D}'$ of the space $\mathscr{D}$ is the topological vector space of all continuous, linear, complex valued functionals, $\varphi' : \mathscr{D} \to \boldsymbol{C}^1$, defined on $\mathscr{D}$. This is the space of distributions on $\mathscr{D}$. The topologies on $\mathscr{D}'$ are the weak dual topology σ of pointwise convergence in $\mathscr{D}$ and the strong dual topology β of uniform convergence on every bounded subset of $\mathscr{D}$.*

Definition 2.7 *The dual space $\mathscr{S}'$ of the space $\mathscr{S}$ is the topological vector space of all continuous, linear, complex valued functionals, $\varphi' : \mathscr{S} \to \boldsymbol{C}^1$, defined on $\mathscr{S}$. This is the space of tempered distributions on $\mathscr{S}$. The topologies on $\mathscr{S}'$ are the weak dual topology σ of pointwise convergence in $\mathscr{S}$ and the strong dual topology β of uniform convergence on every bounded subset of $\mathscr{S}$.*

If Ω is an open subset of $\boldsymbol{R}^1$ and dx is the Lebesgue measure then $\mathscr{L}^p(\Omega)$ with $p \in \boldsymbol{R}^1$, $1 \leqq p < \infty$, denotes the set of all measurable, complex valued functions on $\boldsymbol{R}^1$, $f: \boldsymbol{R}^1 \to \boldsymbol{C}^1$, where $\int_\Omega |f(x)|^p dx < \infty$. In particular $\mathscr{L}^1(\Omega)$ is the space of all functions which are locally integrable in Ω.

From this the topological vector space L^p is defined as the quotient space $L^p(\Omega) = \mathscr{L}^p(\Omega)/\{f \in \mathscr{L}^p(\Omega) \mid \int_\Omega |f(x)|^p dx = 0\}$ of equivalence classes of functions which are equivalent modulo the relation "$f = g$ except on a set of measure zero". The topology on L^p is determined by defining the norm $||f||_{L^p} = (\int_\Omega |f(x)|^p dx)^{1/p}$.

The set of all continuous, linear mappings φ' of $L^p(\Omega)$ into $\boldsymbol{C}^1$, $\varphi' : L^p(\Omega) \to \boldsymbol{C}^1 (p \in \boldsymbol{R}^1,\ 1 \leqq p < \infty)$, constitutes the topological vector space $(L^p(\Omega))'$ $(1 \leqq p < \infty)$ dual to $L^p(\Omega)$. The topologies on $(L^p)'$ are the weak dual topology σ of pointwise convergence in L^p and the strong dual topology β of uniform convergence on every bounded subset of the space $L^p(\Omega)$.

If E and F are two topological vector spaces, a natural injection of E into F with a dense image is a continuous, linear mapping, $j : E \to F$, such that $j(E)$ is dense in F. In diagram 2.1

$$\begin{tikzcd} & \mathscr{S} \arrow[dr] & \\ \mathscr{D} \arrow[ur] \arrow[rr] \arrow[dr] & & L^p \\ & \mathscr{C} \arrow[ur] & \end{tikzcd} \qquad (p \in \boldsymbol{R}^1,\ 1 \leqq p < \infty) \qquad \text{(diagram 2.1)}$$

the arrows indicate possible natural injections with dense images.

From all the images being dense it follows by application of the Hahn-Banach theorem that all the transpose mappings, $j' : F' \to E'$, in diagram 2.2 are injective,

$$\begin{tikzcd}[column sep=small, row sep=tiny] & \mathscr{S}' \arrow[dr] & \\ (L^p)' \arrow[ur] \arrow[rr] \arrow[dr] & & \mathscr{D}' \\ & \mathscr{C}' \arrow[ur] & \end{tikzcd} \qquad (p \in \boldsymbol{R}^1,\ 1 \leqq p < \infty) \qquad \text{(diagram 2.2)}$$

Furthermore, if the two dual spaces E' and F' both carry the weak dual topology σ, or both carry the strong dual topology β, then the transpose mapping, $j' : F' \to E'$, is continuous. This result may be applied to diagram 2.2. Accordingly, all of the spaces in the diagram may be considered as subspaces of the space of distributions $\mathscr{D}'(\Omega)$, *i.e.* all are spaces of distributions. The spaces are all dense in $\mathscr{D}'(\Omega)$.

Again, let Ω be an open subset of $\boldsymbol{R}^1$ and let $p \in \boldsymbol{R}^1$, $1 \leqq p < \infty$. Define $p' = \dfrac{p}{p-1}$ for $p > 1$, and $p' = \infty$ for $p = 1$. Let the space $L^{p'}(\Omega)$ carry the norm topology defined above, and let the space $(L^p(\Omega))'$ carry the weak dual topology σ or the strong dual topology β. If a bilinear form on $L^p(\Omega) \times L^{p'}(\Omega)$ is defined by

$$(f, g) \to \langle f, g \rangle = \int_\Omega \overline{f(x)}\, g(x)\, dx, \tag{2.1}$$

where the bar denotes complex conjugation, then it is an important result from the theory of duality between topological vector spaces that an isomorphism exists between the spaces $L^{p'}(\Omega)$ and $(L^p(\Omega))'$, *i.e.* then a linear mapping exists, $i : L^{p'}(\Omega) \to (L^p(\Omega))'$, which is bicontinuous and bijective. From this it follows that the spaces $\mathscr{D}(\Omega)$, $\mathscr{S}(\Omega)$, $\mathscr{C}(\Omega)$, and $L^p(\Omega)$ $(1 < p < \infty)$ may all be considered as subspaces of the space of distributions $\mathscr{D}'(\Omega)$, *i.e.* their elements may be considered as distributions. The spaces of diagram 2.1 are all dense in the space $\mathscr{D}'(\Omega)$.

Furthermore, the restriction of the continuous, linear form, $L^1(\Omega) \to \boldsymbol{C}^1$, defined by

$$\varphi \to \int_\Omega \overline{f(x)}\, \varphi(x)\, dx, \quad f \in L^1(\Omega), \tag{2.2}$$

to the space $\mathscr{D}(\Omega)$ defines f as a distribution in $\mathscr{D}'(\Omega)$. As finally $\mathscr{D}(\Omega)$ is dense in $L^1(\Omega)$ it follows that if f_1 and f_2 are different as elements of $L^1(\Omega)$ then they will also be different when considered as elements of $\mathscr{D}'(\Omega)$, *i.e.* as distributions.

Theorem 2.3 *Let Ω be an open, non-void subset of $\boldsymbol{R}^1$, $\Omega \subseteqq \boldsymbol{R}^1$. Let $p, p' \in \boldsymbol{R}^1$ such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ and $1 \leqq p, p' \leqq \infty$. Let $f \in L^{p'}(\Omega)$ and $\varphi \in L^p(\Omega)$. Define a mapping by*

$$f \to (\varphi \to \langle f, \varphi \rangle = \int_\Omega \overline{f(x)}\,\varphi(x)\,dx).$$

Then, (*i*) *for* $1 \leqq p < \infty$ *and* $1 < p' \leqq \infty$ *the mapping is a bijective isometry of* $L^{p'}(\Omega)$ *onto* $(L^p(\Omega))'$,

and (*ii*) *for* $p = \infty$ *and* $p' = 1$ *the mapping is an injective isometry of* $L^1(\Omega)$ *into* $(L^\infty(\Omega))'$.

If a function f in $L^1(\Omega)$ exists, $f \in L^1(\Omega)$, such that the continuous, linear form (2.2) defines a distribution T_f in $\mathscr{D}'(\Omega)$, $T_f \in \mathscr{D}'(\Omega)$, which by theorem 2.3 may be identified with the function, $f = T_f$, then the distribution T_f is called regular. If a distribution T, $T \in \mathscr{D}'(\Omega)$, is not regular, it is called singular.

The space $\mathscr{D}'(\Omega)$ possesses a linear vector space structure, such that addition of two distributions f_1 and f_2 in $\mathscr{D}'(\Omega)$ is defined by

$$\langle f_1 + f_2, \varphi \rangle = \langle f_1, \varphi \rangle + \langle f_2, \varphi \rangle, \quad \varphi \in \mathscr{D}(\Omega), \tag{2.3}$$

and multiplication of a distribution f in $\mathscr{D}'(\Omega)$ by a complex number λ is defined by

$$\langle f\lambda, \varphi \rangle = \bar{\lambda} \langle f, \varphi \rangle, \qquad \varphi \in \mathscr{D}(\Omega), \tag{2.4}$$

where the bar denotes complex conjugation.

Multiplication as a bilinear, associative operation on two distributions f_1 and f_2 from a distribution space and coinciding with the multiplication of two elements of $L^1(\Omega)$ (*i.e.* two locally integrable functions) in the case of f_1 and f_2 being regular distributions cannot be defined for arbitrary f_1 and f_2.[3)]

However, the multiplication defined by

$$\varphi \to \alpha\varphi, \qquad \alpha \in \mathscr{C}^\infty(\Omega), \tag{2.5}$$

where Ω is an open subset of $\boldsymbol{R}^1$, is a continuous, linear mapping of $\mathscr{D}(\Omega)$ into itself, $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. Hence the transpose mapping is a continuous, linear mapping of $\mathscr{D}'(\Omega)$ into itself, $\mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, and this transpose mapping is adopted as the definition of multiplication of a distribution in $\mathscr{D}'(\Omega)$ by the function α, $\alpha \in \mathscr{C}^\infty(\Omega)$.

Definition 2.6 *If* Ω *is an open subset of* $\boldsymbol{R}^1$, α *is a function in* $\mathscr{C}^\infty(\Omega)$, $\alpha \in \mathscr{C}^\infty(\Omega)$, *and* T *is a distribution in* $\mathscr{D}'(\Omega)$, *then multiplication of the distribution* T *with the multiplier* α *gives the distribution* αT *defined by* $\langle \alpha T, \varphi \rangle = \langle T, \bar{\alpha}\varphi \rangle$.

3) This is a result due to Schwartz. See ref. (14).

The sets of multipliers in the test function spaces possess linear vector space structures.

Theorem 2.4 *Multipliers in the space* $\mathscr{D}(\Omega)$ *are all infinitely differentiable functions of arbitrary support. Multipliers in the space* $\mathscr{S}(\Omega)$ *are all infinitely differentiable functions s of slow increase, i.e. for which* $|s^{(k)}(x)| = O(|x|^l)$, *where k and l are integers,* $k, l \geqq 0$.

The operation of differentiation of distributions may be defined in an analogous way.

Let f be a function, $f: \boldsymbol{R}^1 \to \boldsymbol{C}^1$, which is differentiable n times, $n = 0, 1, 2, 3, \ldots$, with continuous derivatives in the open subset Ω of $\boldsymbol{R}^1$, $f \in \mathscr{C}^n(\Omega)$. Hence f is also locally integrable, $f \in L^1(\Omega)$, and, as has been expounded above, if φ is a test function in $\mathscr{D}(\Omega)$ then the continuous, linear mapping

$$\varphi \to \int_\Omega \overline{f(x)}\varphi(x)\,dx, \qquad x \in \boldsymbol{R}^1, \tag{2.6}$$

defines in $\mathscr{D}'(\Omega)$ a distribution which may be identified with the function f.

Suppose a homeomorphic mapping of $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is defined by the formula

$$D^n: f \to D^n f, \qquad n = 0, 1, 2, 3, \ldots, \tag{2.7}$$

D^n denoting the n'th order derivative of f in the function sense. Then, as $\int_\Omega f^{(n)}(x)\varphi(x)\,dx = (-1)^n \int_\Omega f(x)\varphi^{(n)}(x)\,dx$, it is seen that $\langle D^n f, \varphi\rangle = (-1)^n \langle f, D^n\varphi\rangle$. This leads to the following definition.

Definition 2.8 *If* Ω *is an open subset of* $\boldsymbol{R}^1$, *n is an integer,* $n = 0, 1, 2, 3, \ldots$, *and T is a distribution in* $\mathscr{D}'(\Omega)$, *then the distribution* $D^n T$ *is defined by* $\langle D^n T, \varphi\rangle = (-1)^n \langle T, D^n \varphi\rangle$.

Notice that D^n is the transpose of $(-1)^n D^n$.

Subsequently the concept of the support of a distribution T, $\operatorname{supp} T$, will be of importance.

Again, Ω is an open subset of $\boldsymbol{R}^1$.

Definition 2.9 *A distribution T in the space* $\mathscr{D}'(\Omega)$ *is said to vanish in an open subset U of* Ω *if* $\langle T, \varphi\rangle = 0$ *for all test functions* $\varphi \in \mathscr{D}(\Omega)$ *with* $\operatorname{supp}\varphi \subseteqq U$.

Definition 2.10 *The support of a distribution T in* $\mathscr{D}'(\Omega)$ *is denoted by* $\operatorname{supp} T$ *and defined as the complement of the largest open subset of* Ω *in which T vanishes.*

The subspace of $\mathscr{D}'(\Omega)$ which consists of all distributions with support on the non-negative real axis $\boldsymbol{R}^1_+ = \{\sigma \in \boldsymbol{R}^1 | 0 \leqq \sigma < \infty\}$ is denoted by $\mathscr{D}'_+ \subset \mathscr{D}'(\Omega)$.

If λ is a complex number, $\lambda \in \boldsymbol{C}^1$, such that $\mathrm{Re}\,\lambda > -1$, if x is real, $x \in \boldsymbol{R}^1$, and if φ is a test function in $\mathscr{D}(\Omega)$, $\varphi \in \mathscr{D}(\Omega)$, then the product function $x^\lambda\varphi$ is locally integrable, $x^\lambda\varphi \in L^1$, and the continuous, linear mapping

$$\varphi \rightarrow \langle x^\lambda_+, \varphi\rangle = \int_0^\infty \overline{x^\lambda}\, \varphi(x)\, dx \tag{2.8}$$

defines the regular distribution $x^\lambda_+ \in \mathscr{D}_+$.

For $\mathrm{Re}\,\lambda \leqq -1$ the limit $\lim\limits_{\varepsilon \to 0+} \int_\varepsilon^\infty x^\lambda\varphi(x)dx$ does not exist, *i.e.* $x^\lambda\varphi \notin L^1$, and therefore the integral may not be used to define a distribution x^λ_+. If, however, a sufficient number of terms are subtracted from the MacLaurin expansion of the test function the integral is rendered convergent. Thus, if $-n < \mathrm{Re}\,\lambda < -(n-1)$, where n is a positive integer, $n = 1, 2, 3, \ldots$, then the integral

$$\langle Pf x^\lambda_+, \varphi\rangle = \int_0^\infty \overline{x^\lambda} \left[\varphi(x) - \sum_{\mu=0}^{n-2} \frac{x^\mu \varphi^{(\mu)}(0)}{\mu!}\right] dx \tag{2.9}$$

converges and is used to define the singular distribution $Pf\, x^\lambda_+ \in \mathscr{D}'_+ \subset \mathscr{D}'(\Omega)$. Distributions of this type are termed pseudofunctions and are characterised by the prefix *Pf*. This procedure of extracting a finite part from a divergent integral by subtracting the terms which cause the integral to diverge was first introduced by HADAMARD[4]), and the result is called the finite part, *Fin. P.*, of the integral.

For $-n < \mathrm{Re}\,\lambda < -(n-1)$, where n is a positive integer, $n = 1, 2, 3, \ldots$, the singular distribution $Pf\, x^\lambda_+$ is defined by

$$\left.\begin{aligned} \langle Pf x^\lambda_+, \varphi\rangle &= Fin.P. \int_0^\infty \overline{x^\lambda} \varphi(x)\, dx \\ &= \int_0^\infty \overline{x^\lambda} \left[\varphi(x) - \sum_{\mu=0}^{n-2} \frac{x^\mu \varphi^{(\mu)}(0)}{\mu!}\right] dx \\ &= \lim_{\varepsilon \to 0+} \left\{\int_\varepsilon^\infty \overline{x^\lambda} \varphi(x)\, dx + \sum_{\mu=0}^{n-2} \frac{\varepsilon^{\lambda+\mu+1} \varphi^{(\mu)}(0)}{\mu!(\lambda+\mu+1)}\right\}, \end{aligned}\right\} \tag{2.10}$$

where $\varphi \in \mathscr{D}(\Omega)$.

For $-1 < \mathrm{Re}\,\lambda$ the continuous, linear mapping

[4]) See ref. (7).

$$\varphi \to \langle Pf\, x_+^\lambda, \varphi\rangle = Fin.\ P. \int_0^\infty \overline{x^\lambda}\varphi(x)\,dx \tag{2.11}$$

may still be used for defining the pseudofunction $Pf\, x_+^\lambda$, but no subtraction in the MacLaurin series is required, so that the distribution defined becomes regular and the notions of *Fin. P.* and *Pf* superfluous,

$$\langle x_+^\lambda, \varphi\rangle = \int_0^\infty \overline{x^\lambda}\varphi(x)\,dx. \tag{2.12}$$

For $\mathrm{Re}\,\lambda = -n$ and $\lambda \neq n$, where n is a positive integer, $n = 1,2,3,\ldots$, the singular distribution $Pf\, x_+^\lambda$ is defined by analytic continuation in the complex λ-plane from $-n < \mathrm{Re}\,\lambda < -(n-1)$. As for $\mathrm{Re}\,\lambda = -n$ the integral $\int_\varepsilon^\infty x^\lambda x^{n-1}\varphi(x)\,dx$ diverges as $\varepsilon \to 0+$, one more term must be included in the series to be subtracted in the integrand in order to obtain the finite part of the integral, *Fin. P.* $\int_0^\infty \overline{x^\lambda}\varphi(x)\,dx$.

Thus, for $\mathrm{Re}\,\lambda = -n$ and $\lambda \neq -n$, where n is a positive integer, $n = 1,2,3,\ldots$, the singular distribution $Pf\, x_+^\lambda$ is defined by

$$\left.\begin{aligned}\langle Pf\, x_+^\lambda, \varphi\rangle &= Fin.\ P. \int_0^\infty \overline{x^\lambda}\varphi(x)\,dx\\ &= \int_0^\infty \overline{x^\lambda}\left[\varphi(x) - \sum_{\mu=0}^{n-2}\frac{x^\mu\varphi^{(\mu)}(0)}{\mu!} - \frac{x^{n-1}\varphi^{(n-1)}(0)}{(n-1)!}\right]dx\\ &= \lim_{\varepsilon\to 0+}\left\{\int_\varepsilon^\infty \overline{x^\lambda}\varphi(x)\,dx + \sum_{\mu=0}^{n-2}\frac{\varepsilon^{\mu+\lambda+1}\varphi^{(\mu)}(0)}{\mu!(\mu+\lambda+1)} + \frac{\varepsilon^{n+\lambda}\varphi^{(n-1)}(0)}{(n-1)!(n+\lambda)}\right\},\end{aligned}\right\} \tag{2.13}$$

where $\varphi \in \mathscr{D}(\Omega)$.

For $\lambda = -1, -2, -3, \ldots$ the above definitions of $Pf\, x_+^{\lambda]}$ do not apply, essentially because in these cases the defining integrals diverge. Considered as functions of $\lambda \in \boldsymbol{C}^1$ the integrals have poles at $\lambda = -1, -2, -3, \ldots$. As φ is in $\mathscr{D}(\Omega)$, $\varphi \in \mathscr{D}(\Omega)$, an integration by parts shows that whereas the integral

$$\int_\varepsilon^\infty \frac{1}{x}\varphi(x)dx = -\ln\varepsilon\,\varphi(0) - \int_\varepsilon^\infty \ln x\,\varphi'(x)dx - o(1) \tag{2.14}$$

diverges as $\varepsilon \to 0+$ because of the term $-\ln\varepsilon\,\varphi(0)$, the integral

$$\left.\begin{aligned}\int_\varepsilon^\infty \frac{1}{x}\varphi(x)\,dx + \ln\varepsilon\,\varphi(0) &= \int_\varepsilon^\infty \frac{1}{x}[\varphi(x) - \varphi(0)u_0(1-x)]dx\\ &= -\int_\varepsilon^\infty \ln x\,\varphi'(x)\,dx - o(1),\\ u_0(x) &= \begin{cases}1 & \text{for} \quad x > 0\\ 0 & \text{for} \quad x < 0\end{cases}\end{aligned}\right\} \tag{2.15}$$

converges as $\varepsilon \to 0+$.

This leads to defining the singular distribution $Pf\left(\frac{1}{x}\right)_+$ corresponding to the case $\lambda = -1$ by

$$\left.\begin{aligned}\langle Pf\left(\frac{1}{x}\right)_+, \varphi\rangle &= \int_0^\infty \frac{1}{x}[\varphi(x) - \varphi(0)\,u_0(1-x)]\,dx \\ &= \lim_{\varepsilon\to 0+}\left\{\int_\varepsilon^\infty \frac{1}{x}\varphi(x)\,dx + \ln\varepsilon\,\varphi(0)\right\}.\end{aligned}\right\} \tag{2.16}$$

It is of importance to consider such effects on a distribution which stem from a change of independent variable in a test function.

Let f be a function, $f: \boldsymbol{R}^1 \to \boldsymbol{C}^1$, which is locally integrable in an open subset Ω of $\boldsymbol{R}^1$, $f \in L^1(\Omega)$. As has been expounded above, if φ is a test function in $\mathscr{D}(\Omega)$ then the continuous, linear mapping

$$\varphi \to \int_\Omega \overline{f(x)}\,\varphi(x)\,dx, \qquad x \in \boldsymbol{R}^1, \tag{2.17}$$

defines in $\mathscr{D}'(\Omega)$ a distribution which may be identified with the function f.

Suppose a homeomorphic mapping of $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is defined by the formula

$$\tau_a : f(x) \to \tau_a f(x) = f(x-a), \qquad a, x \in \boldsymbol{R}^1. \tag{2.18}$$

Then, as $\int_\Omega \overline{f(x-a)}\,\varphi(x)\,dx = \int_\Omega \overline{f(x)}\,\varphi(x+a)\,dx$, it is seen that $\langle \tau_a f, \varphi\rangle = \langle f, \tau_{-a}\varphi\rangle$. This leads to the following definition.

Definition 2.11 *If Ω is an open subset of $\boldsymbol{R}^1$, a is a real number, $a \in \boldsymbol{R}^1$, and T is a distribution in $\mathscr{D}'(\Omega)$, then the distribution $\tau_a T$ is defined by $\langle \tau_a T, \varphi\rangle = \langle T, \tau_{-a}\varphi\rangle$.*

Notice that τ_a is the transpose of τ_{-a}.

Likewise, let a homeomorphic mapping of $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ be defined by the formula

$$\chi_a : f(x) \to \chi_a f(x) = f(ax), \qquad a, x \in \boldsymbol{R}^1. \tag{2.19}$$

Then, as $\int_\Omega \overline{f(ax)}\,\varphi(x)\,dx = \frac{1}{|a|}\int_\Omega \overline{f(x)}\varphi\left(\frac{x}{a}\right)dx$, it is seen that $\langle \chi_a f, \varphi\rangle = \frac{1}{|a|}\langle f, \chi_{a^{-1}}\varphi\rangle$. This motivates the following definition.

Definition 2.12 *If Ω is an open subset of $\boldsymbol{R}^1$, a is a real number, $a \in \boldsymbol{R}^1$, and T is a distribution in $\mathscr{D}'(\Omega)$, then the distribution $\chi_a T$ is defined by*

$$\langle \chi_a T, \varphi\rangle = \frac{1}{|a|}\langle T, \chi_{a^{-1}}\varphi\rangle.$$

Notice that χ_a is the transpose of $\frac{1}{|a|}\chi_{a^{-1}}$.

A linear change of scale, $x \to \alpha x$, $\alpha, x \in \boldsymbol{R}^1$, in the distribution $Pf\left(\frac{1}{x}\right)_+$ defined in (2.16) shows that

$$\left.\begin{aligned} \langle Pf\left(\frac{\alpha}{x}\right)_+, \varphi\rangle &= \int_0^\infty \frac{\alpha}{x}\left[\varphi(x) - \varphi(0)\, u_0\left(1 - \frac{x}{\alpha}\right)\right] dx \\ &= \lim_{\varepsilon \to 0+} \left\{\alpha \int_\varepsilon^\infty \frac{1}{x}\varphi(x)\, dx + \alpha\varphi(0) \ln\left(\frac{\varepsilon}{\alpha}\right)\right\}, \end{aligned}\right\} \tag{2.20}$$

i.e. that

$$\langle Pf\left(\frac{\alpha}{x}\right)_+, \varphi\rangle = \langle \alpha Pf\left(\frac{1}{x}\right)_+ - \alpha \ln \alpha\, \delta, \varphi\rangle. \tag{2.21}$$

In the light of this property it is now possible to define[5] the distribution $Pf\, x_+^\lambda$ also in the case when λ is a negative integer.

Definition 2.13 *Let n be a positive integer, $n = 1, 2, 3, \ldots,$[6] let λ be a complex number, $\lambda \in \boldsymbol{C}^1$, and let φ be a test function in $\mathscr{D}$, $\varphi \in \mathscr{D}$.*

For $-n < Re\,\lambda < -(n-1)$ the distribution $Pf\, x_+^\lambda \in \mathscr{D}'_+$ is defined by

$$\begin{aligned} &\langle Pf\, x_+^\lambda, \varphi\rangle \\ &= Fin.P. \textstyle\int_0^\infty \overline{x^\lambda}\, \varphi(x)\, dx \\ &= \int_0^\infty \overline{x^\lambda}\left[\varphi(x) - \sum_{\mu=0}^{n-2} \frac{x^\mu \varphi^{(\mu)}(0)}{\mu!}\right] dx \\ &= \lim_{\varepsilon \to 0+} \left\{\int_\varepsilon^\infty \overline{x^\lambda}\varphi(x)\, dx + \sum_{\mu=0}^{n-2} \frac{\varepsilon^{\lambda+\mu+1}\varphi^{(\mu)}(0)}{\mu!(\lambda+\mu+1)}\right\}. \end{aligned}$$

For $Re\,\lambda = -n$, $\lambda \neq -n$, the distribution $Pf\, x_+^\lambda \in \mathscr{D}'_+$ is defined by

$$\begin{aligned} &\langle Pf\, x_+^\lambda, \varphi\rangle \\ &= Fin.P. \textstyle\int_0^\infty \overline{x^\lambda}\, \varphi(x)\, dx \\ &= \int_0^\infty \overline{x^\lambda}\left[\varphi(x) - \sum_{\mu=0}^{n-2} \frac{x^\mu \varphi^{(\mu)}(0)}{\mu!} - \frac{x^{n-1}\varphi^{(n-1)}(0)}{(n-1)!}\right] dx \\ &= \lim_{\varepsilon \to 0+} \left\{\int_0^\infty \overline{x^\lambda}\varphi(x)\, dx + \sum_{\mu=0}^{n-2} \frac{\varepsilon^{\mu+\lambda+1}\varphi^{(\mu)}(0)}{\mu!(\mu+\lambda+1)} + \frac{\varepsilon^{n+\lambda}\varphi^{(n-1)}(0)}{(n-1)!(n+\lambda)}\right\}. \end{aligned}$$

[5]) The last part of the definition is due to Güttinger. See ref. (6).
[6]) For $n = 1$ see eq.s (2.11) and (2.12) and accompanying text.

For $\lambda = -n$ *the distribution* $Pf\, x_+^\lambda \in \mathscr{D}'_+$ *is defined by*

$$\begin{aligned}
&\langle Pf\, x_+^\lambda, \varphi\rangle \\
&= Fin.P.\textstyle\int_0^\infty x^{-n}\varphi(x)\,dx \\
&= \int_0^\infty x^{-n}\left[\varphi(x) - \sum_{\mu=0}^{n-2}\frac{x^\mu\varphi^{(\mu)}(0)}{\mu!} - u_0\left(1-\frac{x}{\alpha}\right)\frac{x^{n-1}\varphi^{(n-1)}(0)}{(n-1)!}\right]dx \\
&= \lim_{\varepsilon\to 0+}\left\{\int_\varepsilon^\infty x^{-n}\varphi(x)\,dx + \sum_{\mu=0}^{n-2}\frac{\varepsilon^{\mu-n+1}\varphi^{(\mu)}(0)}{\mu!(\mu-n+1)} + \frac{ln(\varepsilon/\alpha)\varphi^{(n-1)}(0)}{(n-1)!}\right\},
\end{aligned}$$

where $\alpha \in \boldsymbol{R}^1$.

Considered as a function of the complex variable λ, $\lambda \in \boldsymbol{C}^1$, the integral $F(\lambda) = \langle Pf\, x_+^\lambda, \varphi\rangle$ is a complex valued function which is holomorphic everywhere in the finite λ-plane except at the isolated points $\lambda = -1, -2, -3, \ldots$ which constitute a set of simple poles. The function F thus is meromorphic.

The residue at $\lambda = -n$, where n is a positive integer, $n = 1, 2, 3, \ldots$, is found to be

$$\left.\begin{aligned}
&\operatorname*{Res}_{\lambda=-n} F(\lambda) \\
&= \lim_{\lambda\to -n+}(\lambda+n)\int_0^\infty \overline{x^\lambda}\left[\varphi(x) - \sum_{\mu=0}^{n-2}\frac{x^\mu\varphi^{(\mu)}(0)}{\mu!}\right]dx \\
&= \lim_{\lambda\to -n+}(\lambda+n)\left\{\left[\frac{x^{\lambda+1}}{\lambda+1}\left(\varphi(x) - \sum_{\mu=0}^{n-2}\frac{x^\mu\varphi^{(\mu)}(0)}{\mu!}\right)\right]_0^\infty\right. \\
&\qquad - \left[\frac{x^{\lambda+2}}{(\lambda+1)(\lambda+2)}\left(\varphi'(x) - \sum_{\mu=0}^{n-3}\frac{x^\mu\varphi^{(\mu+1)}(0)}{\mu!}\right)\right]_0^\infty \\
&\qquad + \cdots \\
&\quad + (-1)^{n-2}\left[\frac{x^{\lambda+n-1}}{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}(\varphi^{(n-2)}(x) - \varphi^{(n-2)}(0))\right]_0^\infty \\
&\quad + (-1)^{n-1}\left[\frac{x^{\lambda+n}}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}\varphi^{(n-1)}(x)\right]_0^\infty \\
&\quad \left. + (-1)^n\int_0^\infty \frac{x^{\lambda+n}}{(\lambda+1)(\lambda+2)\cdots(\lambda+n)}\varphi^{(n)}(x)\,dx\right\}
\end{aligned}\right\} \quad (2.22)$$

$$\left.\begin{aligned} &= \frac{-1}{(n-1)!}\int_0^\infty \varphi^{(n)}(x)\,dx \\ &= \frac{1}{(n-1)!}\varphi^{(n-1)}(0) \\ &= \frac{(-1)^{n-1}}{(n-1)!}\langle \delta^{(n-1)}, \varphi\rangle. \end{aligned}\right\} \quad (2.22)$$

The Γ-function as a function of the complex variable $\lambda+1$, $\lambda+1 \in \boldsymbol{C}^1$, is meromorphic with isolated, simple poles on the negative, real axis just like the function $F: \lambda \to \langle Pf\, x_+^\lambda, \varphi\rangle$ studied above. For the Γ-function the poles of $\Gamma(\lambda+1)$ are situated at $\lambda = -n$, where n is a positive integer, $n = 1, 2, 3, \ldots$, and the residues are

$$\operatorname*{Res}_{\lambda=-n} \Gamma(\lambda+1) = \frac{(-1)^{n-1}}{(n-1)!} \tag{2.23}$$

For every $\lambda \in \boldsymbol{C}^1$, $\Gamma(\lambda+1) \neq 0$.

The Riesz distributions,[7] first introduced by M. Riesz, are defined from the distributions $Pf\, x_+^\lambda$.

Definition 2.14 *Let $x \in \boldsymbol{R}^1$, $\lambda \in \boldsymbol{C}^1$, and let Γ denote the Γ-function. The Riesz distribution $R_+^\lambda \in \mathscr{D}'_+$ is defined as the distribution* $R_+^\lambda = Pf\dfrac{x_+^\lambda}{\Gamma(\lambda+1)}$.

Because of the properties of the functions $F(\lambda) = \langle Pf\, x_+^\lambda, \varphi\rangle$ and $\Gamma(\lambda+1)$, the integral of the Riesz distribution $\langle Pf\dfrac{x_+^\lambda}{\Gamma(\lambda+1)}, \varphi\rangle = \dfrac{F(\lambda)}{\Gamma(\lambda+1)}$ considered as a function of $\lambda \in \boldsymbol{C}^1$ is holomorphic in the finite λ-plane, *i.e.* $\dfrac{F(\lambda)}{\Gamma(\lambda+1)}$ is an entire function of $\lambda \in \boldsymbol{C}^1$. In particular, the value of the function $\dfrac{F(\lambda)}{\Gamma(\lambda+1)}$ at the points where $F(\lambda)$ and $\Gamma(\lambda+1)$ both have poles may be determined as the quotient of the residues. Therefore

$$\left.\begin{aligned} \langle R_+^{-n}, \varphi\rangle &= \lim_{\lambda\to -n}\langle Pf\frac{x_+^\lambda}{\Gamma(\lambda+1)}, \varphi\rangle \\ &= \lim_{\lambda\to -n}\frac{F(\lambda)}{\Gamma(\lambda+1)} \end{aligned}\right\} \quad (2.24)$$

[7] See ref.s (5), (6), (12), and (14).

$$
\left.\begin{aligned}
&= \frac{\underset{\lambda=-n}{\mathrm{Res}}\ F(\lambda)}{\underset{\lambda=-n}{\mathrm{Res}}\ \Gamma(\lambda+1)} \\
&= \frac{\dfrac{(-1)^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)}{\dfrac{(-1)^{n-1}}{(n-1)!}} \\
&= \varphi^{(n-1)}(0) \\
&= \langle(-1)^{n-1}\delta^{(n-1)}, \varphi\rangle.
\end{aligned}\right\} \qquad (2.24)
$$

Hence the following result.

Theorem 2.5 *For* $\lambda = -n$, *where* $\lambda \in \boldsymbol{C}^1$, *and* n *is a positive integer,* $n = 1, 2, 3, \ldots$, *the Riesz distribution* R_+^λ *is*

$$
R_+^{-n} = \lim_{\lambda \to -n} Pf \frac{x_+^\lambda}{\Gamma(\lambda+1)} = (-1)^{n-1}\delta^{(n-1)}, \quad x \in \boldsymbol{R}^1.
$$

In the particular case of $\lambda = -1$ *the Riesz distribution is the delta distribution,* $R_+^{-1} = \delta$.

3. Properties of the Transformation $\boldsymbol{T}$

In the following a physical system will be characterised by describing the way it responds to some physical stimulus, *i.e.* by describing it as a transformation T of an excitation f from the domain of T, $f \in D(T)$, to a response r in the range of T, $r \in R(T)$. As to the physical nature of f and r no description in more precise terms will be required. The test functions will be defined on $\boldsymbol{R}^1$, $\varphi : \boldsymbol{R}^1 \to \boldsymbol{C}^1$, and though no physical interpretation of the independent variable will be needed the notation and terminology will agree with the case of the variable being real time. It will be assumed subsequently that T is single valued, and therefore the transformation may be written as the mapping $T : f \to r$. In general both $D(T)$ and $R(T)$ will be considered as subsets of the space $\mathscr{D}'(\Omega)$, where Ω is an open subset of $\boldsymbol{R}^1$, *i.e.* both f and r will be assumed to be distributions.

The transformation T will be proposed to have the following six properties.

3. (*i*) *Single valuedness.* To each excitation, $f \in D(T)$, the transformation associates exactly one response, $r \in R(T)$,

$$
T(f) = r. \qquad (3.1)
$$

3. (*ii*) *Linearity.* If α and β are complex numbers, $\alpha, \beta \in \boldsymbol{C}^1$, and f_1 and f_2 are two excitations, $f_1, f_2 \in D(T)$, then

$$T(\alpha f_1 + \beta f_2) = \alpha T(f_1) + \beta T(f_2). \tag{3.2}$$

3. (*iii*) *Stationaryness.* If f is an excitation, $f \in D(T)$, and τ_a is the operator defined by definition 2.11 then the property of stationaryness of T may be stated as $T(\tau_a(f)) = \tau_a(T(f))$, *i.e.* as the commutative property

$$T \circ \tau_a = \tau_a \circ T, \qquad a \in \boldsymbol{R}^1. \tag{3.3}$$

3. (*iv*) *Continuity.* The topologies on $D(T)$ and on $R(T)$ are the topologies induced by the topology on $\mathscr{D}'(\Omega)$. The transformation T is continuous if and only if to each neighbourhood V of $T(0)$ in $R(T)$ there corresponds a neighbourhood U of 0 in $D(T)$ such that $T(U) \subseteqq V$.

There is an important connection between the class of transformations which possess the four properties 3. (*i*)–3. (*iv*) above and the class of transformations which may be represented by a convolution in $\mathscr{D}'(\Omega)$.

If $t \in \boldsymbol{R}^1$, $\tau \in \boldsymbol{R}^1$, and $(t, \tau) \in \boldsymbol{R}^2$, let $\mathscr{D}_t$, $\mathscr{D}_\tau$, and $\mathscr{D}_{t,\tau}$ be the test function spaces $\mathscr{D}$ of all infinitely differentiable functions with compact support defined on $\boldsymbol{R}^1$, $\boldsymbol{R}^1$, and $\boldsymbol{R}^2$, respectively. Denote the corresponding dual spaces by $\mathscr{D}'_t$, $\mathscr{D}'_\tau$, and $\mathscr{D}'_{t,\tau}$, respectively, and let the distributions in the dual spaces be marked by the corresponding indices, *e.g.* U_t, U_τ, and $U_{t,\tau}$, respectively.

Suppose $\varphi : \boldsymbol{R}^1 \times \boldsymbol{R}^1 \to \boldsymbol{C}^1$ is a test function in the space $\mathscr{D}_{t,\tau}$ defined on $\boldsymbol{R}^2$. Then, evidently, the restriction of φ to $\varphi : \boldsymbol{R}^1 \to \boldsymbol{C}^1$ is a test function defined on $\boldsymbol{R}^1$, *e.g.* $\varphi \in \mathscr{D}_t$.

The tensor product of two distributions is a distribution defined as follows.

Definition 3.1 *If $t \in \boldsymbol{R}^1$ and $\tau \in \boldsymbol{R}^1$, U_t and V_τ are two distributions in $\mathscr{D}'$, $U_t \in \mathscr{D}'_t$, and $V_\tau \in \mathscr{D}'_\tau$, and φ is a test function in $\mathscr{D}_{t,\tau}$, then the tensor product $U_t \otimes V_\tau$ of the two distributions U_t and V_τ is a distribution in $\mathscr{D}'_{t,\tau}$, $U_t \otimes V_\tau \in \mathscr{D}'_{t,\tau}$, defined by*

$$\langle U_t \otimes V_\tau, \varphi(t,\tau) \rangle = \langle U_t, \langle V_\tau, \varphi(t,\tau) \rangle \rangle.$$

The following two properties of the tensor product are of importance.

Theorem 3.1 *The commutative rule holds for the tensor product $U_t \otimes V_\tau \in \mathscr{D}'_{t,\tau}$ of $U_t \in \mathscr{D}'_t$ and $V_\tau \in \mathscr{D}'_\tau$, i.e.*

$$\langle U_t \otimes V_\tau, \varphi(t,\tau)\rangle = \langle U_t, \langle V_\tau, \varphi(t,\tau)\rangle\rangle,$$

$$\langle V_\tau \otimes U_t, \varphi(t,\tau)\rangle = \langle V_\tau, \langle U_t, \varphi(t,\tau)\rangle\rangle,$$

and

$$\langle U_t \otimes V_\tau, \varphi(t,\tau)\rangle = \langle V_\tau \otimes U_t, \varphi(t,\tau)\rangle.$$

Theorem 3.1 is Fubini's theorem in distribution theory.

Theorem 3.2 *The support of the tensor product of two distributions U_t and V_τ equals the product of the individual supports of the factors, i.e.*

$$\operatorname{supp}(U_t \otimes V_\tau) = (\operatorname{supp} U_t) \times (\operatorname{supp} V_\tau).$$

Suppose that $t \in \boldsymbol{R}^1$ and $\tau \in \boldsymbol{R}^1$, and that $\varphi \in \mathscr{D}$, $\varphi : \boldsymbol{R}^1 \to \boldsymbol{C}^1$, is a test function defined on $\boldsymbol{R}^1$. From the fact that, for φ defined on $\boldsymbol{R}^1$, φ is a test function in $\mathscr{D}_t$, $\varphi \in \mathscr{D}_t$, *i.e.* φ is infinitely differentiable and has compact support on $\boldsymbol{R}^1$, it cannot be implied that φ extended to $\boldsymbol{R}^2$, $\varphi : \boldsymbol{R}^2 \to \boldsymbol{C}^1$, by $\varphi(t,\tau) = \varphi(t+\tau)$ is a test function in $\mathscr{D}_{t,\tau}$, because even though $\varphi(t,\tau) = \varphi(t+\tau)$ is infinitely differentiable, it does not have compact support on $\boldsymbol{R}^2$. However, if I is a compact subset of $\boldsymbol{R}^2$, $I \subset \boldsymbol{R}^2$, and the function $\alpha \in \mathscr{C}^\infty$, $\alpha : \boldsymbol{R}^2 \to \boldsymbol{R}^1$, is the characteristic function of I, *i.e.* equals one on a neighbourhood of I and equals zero elsewhere, then the product function $\alpha\varphi : \boldsymbol{R}^2 \to \boldsymbol{C}^1$ is a test function, $\alpha\varphi \in \mathscr{D}$, defined on $\boldsymbol{R}^2$.

It is now possible to define the convolution of two distributions in the following cases.

Definition 3.2 *Let $t \in \boldsymbol{R}^1$ and $\tau \in \boldsymbol{R}^1$, let $U_t \in \mathscr{D}'_t$ and $V_\tau \in \mathscr{D}'_\tau$ be two distributions, and let $\varphi \in \mathscr{D}, \varphi : \boldsymbol{R}^1 \to \boldsymbol{C}^1$, be a test function defined on $\boldsymbol{R}^1$. Let I be the intersection $I = (\operatorname{supp} U_t) \cap (\operatorname{supp} V_\tau) \cap (\operatorname{supp} \varphi(t+\tau)) \subset \boldsymbol{R}^2$, and let $\alpha \in \mathscr{C}^\infty$, $\alpha : \boldsymbol{R}^2 \to \boldsymbol{R}^1$, be the characteristic function of the set I.*

If I is compact then the convolution product $U_t * V_\tau$ of the distributions U_t and V_τ is a distribution in $\mathscr{D}'_{t+\tau}$ and is defined by

$$\langle U_t * V_\tau, \varphi(t,\tau)\rangle = \langle U_t \otimes V_\tau, \alpha(t,\tau)\varphi(t+\tau)\rangle.$$

Let the sum of sets $(\operatorname{supp} U_t) + (\operatorname{supp} V_\tau)$ be understood to be the set of points which may be written as the sum of a point t in $\operatorname{supp} U_t$, $t \in \operatorname{supp} U_t$, and a point τ in $\operatorname{supp} V_\tau$, $\tau \in \operatorname{supp} V_\tau$. The following statement holds about the support of a convolution product.

Theorem 3.3 *Let the convolution product $U_t * V_\tau \in \mathscr{D}'_{t+\tau}$ of $U_t \in \mathscr{D}'_t$ and $V_\tau \in \mathscr{D}'_\tau$ be defined. Then $(\operatorname{supp}(U_t * V_\tau)) \subseteqq (\operatorname{supp} U_t) + (\operatorname{supp} V_\tau)$.*

The following theorem is a corollary of theorem 3.1.

Theorem 3.4 *The commutative rule holds for the convolution product* $U_t * V_\tau \in \mathscr{D}'_{t+\tau}$ *of* $U_t \in \mathscr{D}'_t$ *and* $V_\tau \in \mathscr{D}'_\tau$, *i.e.* $U_t * V_\tau = V_\tau * U_t$.

It is of importance to establish sufficient conditions under which the convolution product of two distributions may be defined as above. Only the following two cases will be needed.

Theorem 3.5 *Let* $t \in \boldsymbol{R}^1$ *and* $\tau \in \boldsymbol{R}^1$. *(i) If at least one of he two distributions* $U_t \in \mathscr{D}'_t$ *and* $V_\tau \in \mathscr{D}'_\tau$ *has compact support then the convolution product* $U_t * V_\tau \in \mathscr{D}'_{t+\tau}$ *may be defined. (ii) If both distributions* $U_t \in \mathscr{D}'_t$ *and* $V_\tau \in \mathscr{D}'_\tau$ *have their supports bounded and closed to the left then the convolution product* $U_t * V_\tau \in \mathscr{D}'_{t+\tau}$ *may be defined.*

In both cases the intersection $I = (\operatorname{supp} U_t) \cap (\operatorname{supp} V_\tau) \cap (\operatorname{supp} \varphi(t+\tau))$ *is compact and hence the convolution product well defined by* $\langle U_t * V_\tau, \varphi(t,\tau)\rangle = \langle U_t \otimes V_\tau, \alpha(t,\tau)\varphi(t+\tau)\rangle$.

Such distributions as were encountered in the case (*ii*) above, *i.e.* which have their supports on $\boldsymbol{R}^1$ bounded and closed to the left, are termed right-sided distributions. The set of all right-sided distributions in $\mathscr{D}'$ is denoted by $\mathscr{D}'_R$.

Definition 3.3 $\mathscr{D}'_R \subset \mathscr{D}'$ *is the topological vector space of all right-sided distributions. The topology on* $\mathscr{D}'_R$ *is the inductive limit topology of compact convergence inherited from the space* $\mathscr{D}'$.

The space $\mathscr{D}'_R$ possesses the following important property.

Theorem 3.6 *The space* $\mathscr{D}'_R$ *is a commutative algebra with convolution as rule of composition and with the delta distribution as unit element.*

Notice that the space $\mathscr{D}'_+$ is a particular instance of a space $\mathscr{D}'_R$. Thus, from theorem 3.3 and theorem 3.6 it follows that the convolution product of two distributions in $\mathscr{D}'_+$ is again a distribution in $\mathscr{D}'_+$.

The transformations T which may be written as convolutions are of particular interest. Suppose that only distributions in $\mathscr{D}'_R$ are considered as excitations, $f \in \mathscr{D}'_R$, and suppose there exists in $\mathscr{D}'_R$ a distribution B, $B \in \mathscr{D}'_R$, such that the response r produced by the excitation f may be written as the convolution

$$r = T(f) = B * f, \qquad f \in \mathscr{D}'_R,\ B \in \mathscr{D}'_R. \tag{3.4}$$

The domain of T is $\mathscr{D}'_R$, $D(T) = \mathscr{D}'_R$, and as the excitation f traverses $D(T) = \mathscr{D}'_R$ the response r also traverses $\mathscr{D}'_R$, *i.e.* the range of T is likewise $\mathscr{D}'_R$, $R(T) = \mathscr{D}'_R$.

It is readily established that if a transformation T is defined by the convolution (3.4) then it possesses the four properties 3. (i)–3. (iv) of single valuedness, linearity, stationarynes, and continuity. The validity of the converse assertion has been demonstrated by SCHWARTZ. For the special case of the excitation being right-sided, $f \in \mathscr{D}'_R$, the proposition states that if a transformation T has the whole subspace $\mathscr{D}'_R$ as its domain, $D(T) = \mathscr{D}'_R$, and possesses the four properties 3. (i)–3. (iv), then it is a convolution transformation over $\mathscr{D}'_R$, *i.e.* then there exists a unique distribution $B \in \mathscr{D}'_R$ such that (3.4) is fulfilled. It was seen above that also the range of T will be $\mathscr{D}'_R$, $R(T) = \mathscr{D}'_R$.

The two propositions for right-sided distributions may be stated as a necessary and sufficient condition for the transformation T to be a convolution transformation over $\mathscr{D}'_R$ as follows.

Theorem 3.7 *A unique, right-sided distribution B, $B \in \mathscr{D}'_R$, exists such that transformation T may be defined as the convolution $r = T(f) = B * f$ for $f \in \mathscr{D}'_R$ if and only if T has $\mathscr{D}'_R$ as its domain, $D(T) = \mathscr{D}'_R$, and the transformation possesses the four properties of single valuedness, linearity, stationaryness, and continuity.*

The transformation T will be proposed to possess the two further properties of passivity and causality. In this connection the following operations on distributions will be required.

Definition 3.4 *Let $U \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$. Then*

(i) $\bar{U} \in \mathscr{D}'$ *is defined by* $\langle \bar{U}, \varphi \rangle = \overline{\langle U, \bar{\varphi} \rangle}$, *where the bar denotes complex conjugation,*

(ii) $\check{U} \in \mathscr{D}'$ *is defined by* $\langle \check{U}, \varphi \rangle = \langle U, \check{\varphi} \rangle$, *where* $\check{\varphi}(t) = \varphi(-t)$,

and (iii) $\hat{U} \in \mathscr{D}'$ *is defined by* $\langle \hat{U}, \varphi \rangle = \langle U, \hat{\varphi} \rangle$, *where* $\hat{\varphi}(t) = \overline{\varphi(-t)}$.

Also the next theorem, which essentially informs that the transpose of convolution in $\mathscr{D}$ with the distribution $\hat{V}$ is a convolution in $\mathscr{D}'$ with the distribution V, is called for.

Theorem 3.8 *Let $U \in \mathscr{D}'_R$, $V \in \mathscr{D}'_R$, and $\varphi \in \mathscr{D}$. Then the convolution product $U * V$ is defined, $U * V \in \mathscr{D}'_R$, and $\langle U * V, \varphi \rangle = \langle U, \hat{V} * \varphi \rangle$.*

Suppose that both the excitation f and the response r are elements in $\mathscr{D}$, $f, r \in \mathscr{D}(\Omega)$, where Ω is an open set in $\boldsymbol{R}^1$. Then the transformation

$$T: f \to r, \qquad f, r \in \mathscr{D}(\Omega), \tag{3.5}$$

is said to be passive if

$$\int_{\Omega}[f(t)\overline{f(t)} - r(t)\overline{r(t)}]dt = \int_{\Omega}[|f(t)|^2 - |r(t)|^2]\,dt \geqq 0. \tag{3.6}$$

Using distribution notation together with definition 3.4 equation (3.6) becomes

$$\langle f,f\rangle - \langle r,r\rangle \geqq 0, \qquad f,r \in \mathscr{D}(\Omega). \tag{3.7}$$

Hence, in the case of $f,r \in \mathscr{D}(\Omega)$ the transformation T is said to be passive if equation (3.7) holds. If only right-sided distributions are admitted as excitations, $D(T) = \mathscr{D}'_R$, and if the transformation T is supposed to have the properties 3.(i)–3.(iv) of single valuedness, linearity, stationaryness, and continuity, then according to theorem 3.7 a unique right-sided distribution B exists, $B \in \mathscr{D}'_R$, such that the transformation may be written as

$$r = T(f) = B*f, \qquad f,r,B \in \mathscr{D}'_R. \tag{3.8}$$

As $f \in \mathscr{D}$ and $r \in \mathscr{D}$ implies that $f \in \mathscr{D}'_R$ and $r \in \mathscr{D}'_R$, respectively, equation (3.7) with $f,r \in \mathscr{D}$ may be restated as

$$\left.\begin{aligned}\langle f,f\rangle - \langle r,r\rangle &= \langle \delta*f,f\rangle - \langle B*f,B*f\rangle\\ &= \langle \delta,f*\hat{f}\rangle - \langle B*\hat{B},f*\hat{f}\rangle\\ &= \langle \delta - B*\hat{B},f*\hat{f}\rangle\\ &\geqq 0, \qquad f,r \in \mathscr{D}.\end{aligned}\right\} \tag{3.9}$$

Here $f \in \mathscr{D}$, and it follows that $\hat{f} \in \mathscr{D}$ and that $f*\hat{f} \in \mathscr{D}$. Distributions with the property of (3.9) carry a special name.

Definition 3.5 *Let Ω be an open subset of $\boldsymbol{R}^1$, let $U \in \mathscr{D}'(\Omega)$, and let $\varphi \in \mathscr{D}(\Omega)$. If $\langle U,\varphi*\hat{\varphi}\rangle \geqq 0$ then the distribution U is called positive semi-definite.*

For transformations T which have $D(T) = \mathscr{D}'_R$ and which may be defined as convolutions by equation (3.8) and which therefore according to theorem 3.7 possess the properties 3.(i)–3.(iv) the property of passivity is stated as follows.

3.(v) *Passivity*. The transformation T, which by equation (3.8) is characterised by the distribution $B \in \mathscr{D}'_R$, possesses the property that

$$\langle \delta - B*\hat{B},\varphi*\hat{\varphi}\rangle \geqq 0, \qquad \varphi \in \mathscr{D}, \tag{3.10}$$

i.e. the distribution $\delta - B*\hat{B}$ is positive semi-definite.

3. (*vi*) *Causality*. Let Ω be an open set in $\boldsymbol{R}^1$, and let $t_0 \in \boldsymbol{R}^1$. The property of causality is expressed by stating that if f_1 and f_2 are two excitations in $\mathscr{D}'(\Omega)$, $f_1, f_2 \in \mathscr{D}'(\Omega)$, such that their difference $f_1 - f_2$ vanishes for $t < t_0$ then this shall imply that $T(f_1 - f_2)$ likewise vanishes for $t < t_0$, *i.e.*

$$\left.\begin{array}{l} \operatorname{supp}(f_1 - f_2) = \{t \in \boldsymbol{R}^1 \,|\, t_0 \leqq t\} \\ \text{implies that} \\ \operatorname{supp} T(f_1 - f_2) \subseteqq \{t \in \boldsymbol{R}^1 \,|\, t_0 \leqq t\}. \end{array}\right\} \quad (3.11)$$

For the particular cases when T may be described as a convolution the following theorem is of importance.

Theorem 3.9 *Let T be a convolution transformation with $D(T) = \mathscr{D}'_R$, such that $r = T(f) = B * f$, where $f, r, B \in \mathscr{D}'_R$. The transformation T is causal if and only if* $\operatorname{supp} B \subseteqq \{t \in \boldsymbol{R}^1 \,|\, 0 \leqq t\}$.

It may be shown that if the transformation T possesses the first five properties 3. (*i*)–3. (*v*) of single valuedness, linearity, stationaryness, continuity, and passivity, then it also possesses the sixth property 3. (*vi*) of causality. In view of theorem 3.7 this assertion may be stated as the following sufficient condition for causality.

Theorem 3.10 *Let T be a convolution transformation with $D(T) = \mathscr{D}'_R$, such that $r = T(f) = B * f$, where $f, r, B \in \mathscr{D}'_R$, and let T be passive, then it is also causal.*

4. Laplace Transformation of the Transformation Equation

By reference to theorem 3.7 a transformation T for which $D(T) = \mathscr{D}'_R$ and which possesses the four properties 3. (*i*)–3. (*iv*) may be completely described as a convolution transformation by

$$r = T(f) = B * f, \qquad f, r, B \in \mathscr{D}'_R. \tag{4.1}$$

This means that if only excitations f which are elements in the space $\mathscr{D}'_R$ considered as a convolution algebra (cfr. theorem 3.6) are admitted, $f \in \mathscr{D}'_R$, then T is completely characterised by the element $B \in \mathscr{D}'_R$, *i.e.* by giving the t-representation of the transformation T. If the distributions f, r, and B of (4.1) all possess Laplace transforms then the transformation T and the corresponding physical system are completely characterised by the

Laplace transform of B, $\mathscr{L}B = S(p)$, *i.e.* by giving the p-representation of the transformation T. The function $S: \boldsymbol{C}^1 \supseteqq D(S) \to R(S) \subseteqq \boldsymbol{C}^1$ is a complex valued function of a complex variable, $p \in \boldsymbol{C}^1$. It is called the system function[8]. The distribution B in (4.1) is called the t-representation of the system function.

Definition 4.1 *Let Γ be a subset of $\boldsymbol{R}^1$, and let $x \in \boldsymbol{R}^1$ and $\sigma \in \boldsymbol{R}^1$. The topological vector space $\mathscr{S}'_x(\Gamma)$ is the space of all distributions $U_x \in \mathscr{D}'_x$ such that $e^{-\sigma x} U_x \in \mathscr{S}'_x$ for $\sigma \in \Gamma$. A sequence $\{U_\nu\}$, $\nu = 1, 2, 3, \ldots$ is defined to converge to the limit U in the topology on $\mathscr{S}'_x(\Gamma)$ if and only if for every $\sigma \in \Gamma$ the sequence $\{e^{-\sigma x} U_\nu\}$, $\nu = 1, 2, 3, \ldots$ converges to the limit $e^{-\sigma x} U$ in the weak dual topology σ on $\mathscr{S}'_x$.*

It may be shown that Γ is convex and hence in the present case of $\Gamma \subseteqq \boldsymbol{R}^1$, if Γ is not empty, Γ is an interval on the real axis, finite, semi-infinite, or infinite.

A sufficient condition for the Laplace transform of a distribution to exist is that the distribution is an element of the space $\mathscr{S}'(\Gamma)$.

Definition 4.2 *Let Γ be a convex subset of $\boldsymbol{R}^1$, let $x, \sigma, \omega \in \boldsymbol{R}^1$, let $B_x \in \mathscr{D}'_x$, and let $\alpha \in \mathscr{C}^\infty$, $\alpha: \boldsymbol{R}^1 \to \boldsymbol{R}^1$, be the characteristic function of* $\operatorname{supp} B_x$. *If $B_x \in \mathscr{S}'_x(\Gamma)$ then the Laplace transform of B, $\mathscr{L}B$, is defined as*

$$\mathscr{L}B = \langle B_x, \alpha(x) e^{-\sigma x} e^{-i\omega x} \rangle = \langle B_x, \alpha(x) e^{-px} \rangle = S(p),$$

where $\quad p \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1.$

The following theorem establishes a very important connection between the properties of a distribution B and the properties of its Laplace transform. In the theorem the same notation is used as in the definition 4.2 above.

Theorem 4.1 *Let Γ be an open, convex subset of $\boldsymbol{R}^1$. If $S: \boldsymbol{C}^1 \supset \Gamma + i\boldsymbol{R}^1 \to \boldsymbol{C}^1$ is a function which is holomorphic in the open strip $\Gamma + i\boldsymbol{R}^1$, and if $|S(p)|$, where $p = \sigma + i\omega \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$, on each compact subset K of Γ, $K \subset \Gamma$, is majorised by a polynomial in $|\omega|$ depending on K, $|S(p)| \leqq \mathscr{P}_K(|\omega|)$, then a unique distribution B in $\mathscr{S}'(\Gamma)$ exists, $B \in \mathscr{S}'(\Gamma)$, such that $\mathscr{L}B = S(p)$. Conversely, if B is a distribution in $\mathscr{S}'(\Gamma)$, $B \in \mathscr{S}'(\Gamma)$, then a unique function exists, $S: \boldsymbol{C}^1 \supset \Gamma + i\boldsymbol{R}^1 \to \boldsymbol{C}^1$, which is holomorphic in the open strip $\Gamma + i\boldsymbol{R}^1$, which on each compact subset K of Γ, $K \subset \Gamma$, is majorised by a polynomial in $|\omega|$, $|S(p)| \leqq \mathscr{P}_K(|\omega|)$, and which is the Laplace transform of B, $S(p) = \mathscr{L}B$.*

[8]) See ref.s (10) and (11).

If a complex valued function is known to be the Laplace transform of a distribution then information about the support of the distribution may be obtained from the following theorem.

Theorem 4.2. *Let* $x_0 \in \boldsymbol{R}^1$, *and let* Γ *be the open, convex half line* $\Gamma = \{\sigma \in \boldsymbol{R}^1 \mid x_0 < \sigma\} \subset \boldsymbol{R}^1$. *If* $S: \boldsymbol{C}^1 \supset \Gamma + i\boldsymbol{R}^1 \to \boldsymbol{C}^1$ *is a function which is holomorphic in the open half plane* $\Gamma + i\boldsymbol{R}^1$, *and if* $|S(p)|$, *where* $p = \sigma + i\omega \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$, *on each compact subset* K *of* Γ, $K \subset \Gamma$, *is majorised by the product of a function* $e^{-x_0\sigma}$ *and a polynomial in* $|\omega|$ *depending on* K, $|S(p)| \leqq e^{-x_0\sigma}\mathscr{P}_K(|\omega|)$, *then a unique, right-sided distribution* B *in* $\mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$ *exists*, $B \in \mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$, *such that its support is bounded to the left by* x_0, $\operatorname{supp} B_x \subseteq \{\sigma \in \boldsymbol{R}^1 \mid x_0 \leqq \sigma\} = \overline{\Gamma}$, *and such that* $\mathscr{L}B = S(p)$. *Conversely, if* B *is a right-sided distribution in* $\mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$, $B \in \mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$, *such that its support is bounded to the left by* x_0, $\operatorname{supp} B_x \subseteq \{\sigma \in \boldsymbol{R}^1 \mid x_0 \leqq \sigma\} = \overline{\Gamma}$, *then a unique function exists*, $S: \boldsymbol{C}^1 \supset \Gamma + i\boldsymbol{R}^1 \to \boldsymbol{C}^1$, *which is holomorphic in the open half plane* $\Gamma + i\boldsymbol{R}^1$, *which on each compact subset* K *of* Γ, $K \subset \Gamma$, *is bounded by the product of a function* $e^{-x_0\sigma}$ *and a polynomial in* $|\omega|$ *depending on* K, $|S(p)| \leqq e^{-x_0\sigma}\mathscr{P}_K(|\omega|)$, *and which is the Laplace transform of* B, $S(p) = \mathscr{L}B$.

For transformations T which may be written as convolution transformations according to theorem 3.7 the following theorem is important.

Theorem 4.3 *Let* Γ *be an open, convex subset of* $\boldsymbol{R}^1$, *and let* $f, r, B \in \mathscr{S}'(\Gamma)$. *The Laplace transforms* $\mathscr{L}f = \mathscr{F}(p)$, $\mathscr{L}B = S(p)$, *and* $\mathscr{L}r = \mathscr{R}(p)$, *where* $p \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$, *are holomorphic functions in the open strip* $\Gamma + i\boldsymbol{R}^1$. *If* $r = B * f$ *then also* $\mathscr{R}(p) = S(p)\mathscr{F}(p)$.

The particular instances of Laplace transforms given in the next two theorems will be required.

Theorem 4.4 *The Laplace transform of the* δ*-functional is one*, $\mathscr{L}\delta = 1$.

Theorem 4.5 *Let* $\lambda, p \in \boldsymbol{C}^1$. *The Laplace transform of the Riesz distribution* $Pf\dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathscr{S}' \cap \mathscr{D}'_R$ *is* $\mathscr{L}Pf\dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)} = \dfrac{1}{p^\lambda}$.

Certain symmetry properties in the distribution B are reflected in the properties of its Laplace transform $\mathscr{L}B$.

Theorem 4.6 *Let* Γ *be an open, convex subset of* $\boldsymbol{R}^1$, *let* $B \in \mathscr{S}'(\Gamma)$, *and let* $\mathscr{L}B = S(p)$, *where* $p \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$.

Then (*i*) $\mathscr{L}\overline{B} = \overline{S(\overline{p})}$,
(*ii*) $\mathscr{L}\check{B} = S(-p)$
and (*iii*) $\mathscr{L}\hat{B} = \overline{S(-\overline{p})}$

The distribution B which characterises a convolution transformation T in (4.1) will be supposed to be real. Compare with the definition 3.4 of the complex conjugate of a distribution.

Definition 4.3 *Let Ω be an open subset of $\boldsymbol{R}^1$, and let B be a distribution in $\mathscr{D}'(\Omega)$, $B \in \mathscr{D}'(\Omega)$. B is defined to be a real distribution if and only if $B = \overline{B}$ on Ω.*

The Laplace transforms of distributions which are real possess the following property.

Theorem 4.7 *Let Γ be an open, convex subset of $\boldsymbol{R}^1$, let $B \in \mathscr{S}'(\Gamma)$, and let $\mathscr{L}B = S(p)$, where $p \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$. The distribution B is real, $B = \overline{B}$, if and only if $S(p) = \overline{S(\overline{p})}$.*

The following result is a consequence of the theorems 4.4, 4.5, and 4.6.

Theorem 4.8 *Let B be as in theorems 4.5 and 4.6. Then $\mathscr{L}(\delta - B * \hat{B}) = 1 - S(p)\overline{S(-\overline{p})}$. If furthermore B is real, $B = \overline{B}$, then $\mathscr{L}(\delta - B * \hat{B}) = 1 - S(p)S(-p)$.*

Finally, in order to state the theorem of Bochner and Schwartz[9] which will be needed subsequently the concept of a positive, tempered measure must be introduced.

Definition 4.4 *Let Ω be an open subset of $\boldsymbol{R}^1$, and let φ be a test function in the space $\mathscr{C}(\Omega)$, such that $\varphi(x) \in \boldsymbol{R}^1$ and $\varphi(x) \geqq 0$ for all $x \in \Omega$. Let μ be a measure in the dual space, $\mu \in \mathscr{C}'(\Omega)$. The measure μ is defined to be positive if and only if $\langle \mu, \varphi \rangle \geqq 0$ for all such φ.*

Definition 4.5 *Let $r, A \in \boldsymbol{R}^1$, let Ω be an open subset of $\boldsymbol{R}^1$, and let the measure $\mu \in \mathscr{C}'(\Omega)$ be positive. The positive measure μ is defined to be a positive, tempered measure if and only if an integer l exists, $l \geqq 0$, such that $\int_{r \leqq A} |d\mu| = O(A^l)$ as $A \to \infty$.*

As indicated in 3.(v), if the transformation T is a convolution, so that $r = T(f) = B * f$, then T is defined to be passive if the distribution $\delta - B * \hat{B}$

[9]) The theorem, initially stated by Bochner, ref. (1), has been generalised by Schwartz, ref.s (5) and (14).

is positive semi-definite. In this connection the following theorem due to BOCHNER and SCHWARTZ is of importance as it presents a criterion to establish if a distribution is positive semi-definite.

Theorem 4.9 *Let Γ be an open, convex subset of $\boldsymbol{R}^1$, $\Gamma \subset \boldsymbol{R}^1$, let $B \in \mathscr{S}'(\Gamma)$, and let $\mathscr{L}B = S(p)$, where $p = \sigma + i\omega \in \Gamma + i\boldsymbol{R}^1 \subset \boldsymbol{R}^1 + i\boldsymbol{R}^1 = \boldsymbol{C}^1$. The distribution B is positive semi-definite if and only if the restriction of the Laplace transform to the imaginary axis $i\boldsymbol{R}^1$, $\mathscr{L}B|_{\sigma=0} = S(i\omega)$, is a positive, tempered measure.*

5. The General Debye Function as System Function

When measurement of dielectric relaxation of physical systems is carried out the p-representation of the transformation T, which maps the excitation E, the electric field in the dielectric, into the response D, the displacement field in the dielectric, is determined as the system function

$$S(p) = \frac{\varepsilon(p) - \varepsilon_\infty}{\varepsilon_s - \varepsilon_\infty}, \qquad p \in \boldsymbol{C}^1, \tag{5.1}$$

where p is the complex frequency, $p = \sigma + i\omega$, $\varepsilon(p)$ is the complex dielectric constant, and where ε_s and ε_∞ are the limit values $\varepsilon_s = \lim\limits_{\omega \to 0+} \varepsilon(\sigma + i\omega)$ and $\varepsilon_\infty = \lim\limits_{\omega \to \infty} \varepsilon(\sigma + i\omega)$, both of which are real.

The assumption that the response of the dielectric, the displacement field D, displays exponential decay to a delta functional excitation in the electric field E is equivalent to the assumption that the system function S is of the form

$$S(p) = \frac{\varepsilon(p) - \varepsilon_\infty}{\varepsilon_s - \varepsilon_\infty} = \frac{1}{1 + p\tau_0}, \qquad p \in \boldsymbol{C}^1, \qquad \tau_0 \in \boldsymbol{R}^1. \tag{5.2}$$

Here τ_0 is a positive number, $\tau_0 > 0$, the relaxation time.

The function

$$S(p) = \frac{1}{1 + p\tau_0} \tag{5.3}$$

is called the Debye function. It is holomorphic in the open half plane $\Gamma + i\boldsymbol{R}^1$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | -1/\tau_0 < \sigma\}$, and it maps the half line L_{σ_0} in the p-plane

$$L_{\sigma_0} = \{\sigma_0 + i\omega \in \boldsymbol{R}^1 + i\boldsymbol{R}^1 | 0 \leqq \omega\}, \qquad \sigma_0 > \frac{-1}{\tau_0}, \tag{5.4}$$

into the semi circle in the $S(p)$-plane

$$\left. \begin{aligned} A_{\sigma_0} = \Bigg\{ S(p) \in \boldsymbol{C}^1 \Bigg| |S(p) - \frac{1}{2(1+\sigma_0\tau_0)}| = \frac{1}{2(1+\sigma_0\tau_0)}, \\ \operatorname{Im} S(p) \leqq 0 \Bigg\}, \qquad \sigma_0 > \frac{-1}{\tau_0}, \end{aligned} \right\} \tag{5.5}$$

the Cole-Cole semi circle. A special instance is the case of $\sigma_0 = 0$, when

$$S(i\omega) = \frac{\varepsilon(i\omega) - \varepsilon_\infty}{\varepsilon_s - \varepsilon_\infty} = \frac{1}{1 + i\omega\tau_0}, \tag{5.6}$$

and the half line L_0 is mapped into the semi circle A_0,

$$A_0 = \{S(p) \in \boldsymbol{C}^1 | |S(i\omega) - \tfrac{1}{2}| = \tfrac{1}{2}, \ \operatorname{Im} S(p) \leqq 0\}. \tag{5.7}$$

In many cases, however, *e.g.* of dielectric systems, there has been reported experimental evidence that the half line L_0 in the p-plane is not mapped into a semi circle A_0 in the $S(p)$-plane, but rather into various forms of continuous arcs, circular arcs, skew symmetric arcs, etc. All of this evidence indicates that the primary assumption of the response of the dielectric system displaying an exponential decay characterised by the sole parameter τ_0, the relaxation time, to a delta functional excitation, cannot hold in general.

This has led to attempts to alter the system function S to a form justified by its compatibility with experimental observations, *i.e.* a phenomenological form.

The functions

$$S(p) = \frac{1}{1 + (p\tau_0)^{1-\alpha}}, \ p \in \boldsymbol{C}^1, \ \alpha, \tau_0 \in \boldsymbol{R}^1 \ (\text{Cole-Cole}),^{10)} \tag{5.8}$$

and

$$S(p) = \frac{1}{[1 + p\tau_0]^\beta}, \ p \in \boldsymbol{C}^1, \ \beta, \tau_0 \in \boldsymbol{R}^1 \ (\text{Davidson-Cole}),^{11)} \tag{5.9}$$

where $0 \leqq \alpha, \beta \leqq 1$, both have been used to characterise dielectric systems. A few years ago the still more general function

[10]) See ref. (3).
[11]) See ref. (4).

$$S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}, \; p \in \boldsymbol{C}^1, \; \alpha, \beta, \tau_0 \in \boldsymbol{R}^1 \; (\text{Havriliak-Negami}),^{12)} \quad (5.10)$$

where again $0 \leqq \alpha, \beta \leqq 1$, was proposed as system function to characterise certain polymer dielectric systems, the proposal being justifiable by the ensuing agreement with the experimental observations.

The function (5.10) is called the general Debye function. It is a complex valued function of a complex variable, $S: \boldsymbol{C}^1 \supseteqq D(S) \to R(S) \subseteqq \boldsymbol{C}^1$. In contrast to the Debye function (5.3), $S(p) = \frac{1}{1 + p\tau_0}$, which as its sole singularity has a first order pole at $p = -1/\tau_0$, the general Debye function (5.10), $S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}$, possesses singularities which may be essential.

5. (*i*) $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1, \quad \alpha = 1, \quad 0 \leqq \beta \leqq 1, \quad \tau_0 > 0.$

In this case the function (5.10) becomes

$$S(p) = \frac{1}{2^\beta}, \tag{5.11}$$

which is a holomorphic function in the entire p-plane, and $|S(p)|$ is bounded by

$$\tfrac{1}{2} \leqq |S(p)| = \frac{1}{2^\beta} \leqq 1. \tag{5.12}$$

The domain of holomorphy includes the open half plane $\Gamma + i\boldsymbol{R}^1$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | 0 < \sigma\}$, and it follows from theorem 4.2 that a unique right-sided distribution B_t exists, $B_t \in \mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$, such that $\mathscr{L}B = \frac{1}{2^\beta}$, and which has its support bounded to the left at $t = 0$, $\operatorname{supp} B_t \subseteqq \{t \in \boldsymbol{R}^1 | 0 \leqq t\}$. An application of theorem 4.7 shows that B_t is real. It is seen immediately that $B = \frac{1}{2^\beta}\delta$.

5. (*ii*) $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1, \quad 0 \leqq \alpha < 1, \quad 0 \leqq \beta \leqq 1, \quad \tau_0 > 0.$

In this case the function (5.10) is many valued and has discrete branch points p_k, which may be dense on the circle $|p| = \frac{1}{\tau_0}$, and which are situated at

[12]) See ref.s (8) and (9).

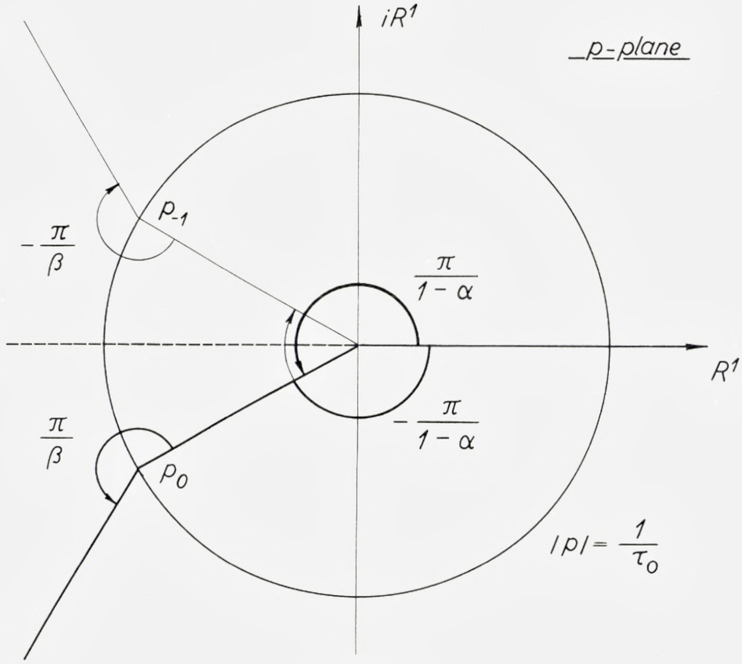

Fig. 5.1.

$$p_k = \frac{1}{\tau_0}\exp\left[i\pi\frac{1+2k}{1-\alpha}\right],\ k = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{5.13}$$

In addition, a branch point is situated at $p = 0$.

Each of the branch points p_k is of infinitely high order if and only if β is irrational. The branch points p_k are dense on the circle $|p| = \frac{1}{\tau_0}$, and the branch point $p = 0$ is of infinitely high order if and only if α is irrational. However, the branch points p_k are situated on different sheets of the Riemann surface. In fig. 5.1 the sheet of the Riemann surface corresponding to the principal branch of the function (5.10), *i.e.* corresponding to the branch which contains the set $\{S(p) \in \boldsymbol{C}^1 \,|\, \operatorname{Arg} S(p) = 0\}$, is indicated. The sheet contains the open half plane $\{p \in \boldsymbol{C}^1 \,|\, 0 < |p| \wedge |\operatorname{Arg} p| < \pi/2\}$, which may be continued to the sector $\{p \in \boldsymbol{C}^1 \,|\, 0 < |p| \wedge |\operatorname{Arg} p| < \pi/(1-\alpha)\}$ and even further, compare fig. 5.1. The mapping

$$S\colon p \to \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta} \quad \text{(principal branch)} \tag{5.14}$$

from this sheet of the Riemann surface to the principal branch of (5.10) is holomorphic.

Let a branch cut in the p-plane be introduced along the negative, real axis from $p = 0$ to the point at infinity. Then the mapping (5.14) is holomorphic in the entire p-plane except on the negative, real axis, the set $\{\sigma \in \boldsymbol{R}^1 | \sigma \leqq 0\}$. As this domain of holomorphy includes the open half plane $\Gamma + i\boldsymbol{R}^1$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | 0 < \sigma\}$ and as

$$|S(p)| \leqq 1, \tag{5.15}$$

it follows again from the theorem 4.2 that a unique, right-sided distribution B_t exists, $B_t \in \mathscr{S}'(\Gamma) \cap \mathscr{D}'_R$, such that $\mathscr{L}B = S(p)$, where $S(p)$ is understood to be the principal branch of the function (5.14), and which distribution has its support bounded to the left at $t = 0$, $\operatorname{supp} B_t \subseteq \{t \in \boldsymbol{R}^1 | 0 \leqq t\}$. As $S(\bar{p}) = \overline{S(p)}$, it follows from theorem 4.7 that B_t is real.

Analogous considerations may be applied to the function $S \circ n$, where n is the mapping

$$n: p \to -p, \qquad p \in \boldsymbol{C}^1. \tag{5.16}$$

If a branch cut in the p-plane is introduced along the positive, real axis from $p = 0$ to the point at infinity then the function $S \circ n$ is holomorphic in the entire p-plane except on the positive, real axis, the set $\{\sigma \in \boldsymbol{R}^1 | 0 \leqq \sigma\}$. Consequently the function $1 - S(p)S(-p)$ is holomorphic in the two open half planes $\boldsymbol{R}^1 + i\Gamma$ and $\boldsymbol{R}^1 - i\Gamma$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | 0 < \sigma\}$. In particular, the function $1 - S(p)S(-p)$ is holomorphic on the imaginary axis with the point $p = 0$ excluded, $i\boldsymbol{R}^1 \setminus \{0\}$. At the point $p = 0$ the function $1 - S(p)S(-p)$ is continuous.

The restriction of the function $1 - S(p)S(-p)$ to the imaginary axis $i\boldsymbol{R}^1$ is

$$1 - S(i\omega)S(-i\omega) = 1 - |S(i\omega)|^2 \in \boldsymbol{R}^1 \tag{5.17}$$

for which

$$0 \leqq 1 - |S(i\omega)|^2 \leqq 1, \qquad 0 \leqq \alpha, \beta \leqq 1, \qquad 0 < \tau_0. \tag{5.18}$$

From theorem 4.9 (Bochner-Schwartz) it follows that the distribution $\delta - B * \hat{B}$, where $\mathscr{L}B = \dfrac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}$ (principal branch), is positive semi-definite.

The situation is summarised in the following theorem.

Theorem 5.1 *Let T be a convolution transformation with $D(T) = \mathscr{D}'_R$, such that $r = T(f) = B * f$, where $f, r, B \in \mathscr{D}'_R$. Let $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1$, with $0 \leqq \alpha, \beta \leqq 1$, and $0 < \tau_0$, and let $\mathscr{L}B = S(p)$, where $S(p)$ is the principal branch of the function $S(p) = \dfrac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}$.*

Then the transformation $T: f \to r$ possesses the six properties of (i) single valuedness, (ii) linearity, (iii) stationaryness, (iv) continuity, (v) passivity, and (vi) causality.

6. The *t*-Representation of the General Debye Function

According to the results in sections 5.(*i*) and 5.(*ii*) the function

$$S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta} \qquad \text{(principal branch)}, \tag{6.1}$$

where $p \in \boldsymbol{C}^1$, and $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1$ with $0 \leqq \alpha, \beta \leqq 1$, $0 < \tau_0$, is holomorphic in the open half plane $\Gamma + i\boldsymbol{R}^1$ where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | 0 < \sigma\}$. In the rest of section 6 when referring to the function (6.1) only the principal branch is considered.

If $\dfrac{1}{\tau_0} < |p|$ and hence also if p is in the open half plane $\Gamma_1 + i\boldsymbol{R}^1$, $p \in \Gamma_1 + i\boldsymbol{R}^1$, where $\Gamma_1 = \left\{\sigma \in \boldsymbol{R}^1 \middle| \dfrac{1}{\tau_0} < \sigma\right\}$, the function (6.1) may be expanded in an infinite binomial series. For $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1$, with $0 \leqq \alpha, \beta \leqq 1$, and $0 < \tau_0$, the expansion is

$$\left.\begin{aligned} S(p) &= \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta} \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(1-\beta)}{n!\,\Gamma(1-\beta-n)} \cdot \frac{1}{(p\tau_0)^{(1-\alpha)(\beta+n)}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{1}{(p\tau_0)^{(1-\alpha)(\beta+n)}}, \end{aligned}\right\} \tag{6.2}$$

where Γ denotes the Γ-function and where the identity

$$\Gamma(\beta)\Gamma(1-\beta) = (-1)^n\Gamma(\beta+n)\Gamma[1-(\beta+n)], \; n = 0, \pm 1, \pm 2, \ldots \tag{6.3}$$

has been used. If p is in the intersection of $\frac{1}{\tau_0} < |p|$ and the open half plane $\Gamma + i\boldsymbol{R}^1$, *i.e.* for $p \in \left\{ z \in \boldsymbol{C}^1 \,\middle|\, \frac{1}{\tau_0} < |z| \wedge z \in \Gamma + i\boldsymbol{R}^1 \right\}$ where $\Gamma = \{\sigma \in \boldsymbol{R}^1 \,|\, 0 < \sigma\}$, the series (6.2) is (C, 1) summable on the set $\{(\alpha, \beta) \in \boldsymbol{R}^2 \,|\, 0 \leqq \alpha \leqq 1 \wedge 0 \leqq \beta \leqq 1\}$ and uniformly convergent on the subset $\{(\alpha, \beta) \in \boldsymbol{R}^2 \,|\, 0 \leqq \alpha < 1 \wedge 0 \leqq \beta < 1\}$.

The function (6.1) $S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}$ (principal branch) is the analytic continuation to the open half plane $\Gamma + i\boldsymbol{R}^1$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 \,|\, 0 < \sigma\}$, of the function which for $p \in \left\{ z \in \boldsymbol{C}^1 \,\middle|\, \frac{1}{\tau_0} < |z| \wedge z \in \Gamma + i\boldsymbol{R}^1 \right\}$ is represented by the infinite series (6.2).

From theorem 4.5 it follows that the n'th term of the series (6.2) is the Laplace transform of the Riesz distribution $(B_t)_n = \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta + n)}{\Gamma(\beta)} \cdot Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}$, *i.e.*

$$\left.\begin{aligned} \mathscr{L}(B_t)_n &= \mathscr{L}\frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]} \\ &= \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{1}{(p\tau_0)^{(1-\alpha)(\beta+n)}}. \end{aligned}\right\} \quad (6.4)$$

For $p \in \left\{ z \in \boldsymbol{C}^1 \,\middle|\, \frac{1}{\tau_0} < |z| \wedge z \in \Gamma + i\boldsymbol{R}^1 \right\}$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 \,|\, 0 < \sigma\}$, termvise application of the result (6.4) to the series (6.2) yields the infinite series of Riesz distributions

$$\left.\begin{aligned} B_t &= \sum_{n=0}^{\infty} (B_t)_n \\ &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}. \end{aligned}\right\} \quad (6.5)$$

According to theorem 4.2 the function (6.1) $S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^\beta}$ (principal branch) determines a unique distribution B_t, which is right-sided and is an element of $\mathscr{S}'_t(\Gamma)$ where $\Gamma = \{\sigma \in \boldsymbol{R}^1 \,|\, 0 < \sigma\}$, $B_t \in \mathscr{D}'_R \cap$

$\mathscr{S}'_t(\Gamma)$, which has its support bounded to the left at $t = 0$, $\operatorname{supp} B_t \subseteq \{t \in \boldsymbol{R}^1 | 0 \leqq t\}$, and which is such that $\mathscr{L} B_t = S(p)$, where $S(p)$ is the function (6.1). The series (6.5) does not converge in the topology on $\mathscr{S}'_t(\Gamma)$, but in the coarser topology on $\mathscr{S}'_t(\Gamma_1)$, where $\Gamma_1 = \left\{\sigma \in \boldsymbol{R}^1 \middle| \frac{1}{\tau_0} < \sigma\right\}$, the series (6.5) is (C, 1) summable on the set $\{(\alpha, \beta) \in \boldsymbol{R}^2 | 0 \leqq \alpha \leqq 1 \wedge 0 \leqq \beta \leqq 1\}$ and convergent on the subset $\{(\alpha, \beta) \in \boldsymbol{R}^2 | 0 \leqq \alpha < 1 \wedge 0 \leqq \beta < 1\}$.

It is of interest to indicate the series for B_t and for $S(p)$ in the four special cases of $(\alpha, \beta) = (1,1)$, $(1,0)$, $(0,1)$, and $(0,0)$. In all four cases $p \in \left\{z \in \boldsymbol{C}^1 \middle| \frac{1}{\tau_0} < |z| \wedge z \in \Gamma + i\boldsymbol{R}^1\right\}$, where $\Gamma = \{\sigma \in \boldsymbol{R}^1 | 0 < \sigma\}$, and $0 < \tau_0$.

6. (i) $\alpha = 1$, $\beta = 1$.

$$\left.\begin{aligned} B_t &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \left.\frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}\right|_{\substack{\alpha=1\\ \beta=1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \delta_{t/\tau_0} \\ &= \sum_{n=0}^{\infty} (-1)^n \delta_t \\ &= \tfrac{1}{2} \delta_t . \end{aligned}\right\} \tag{6.6}$$

$$S(p) = \frac{1}{[1 + (p\tau_0)^{1-\alpha}]^{\beta}} = \tfrac{1}{2}. \tag{6.7}$$

6. (ii) $\alpha = 1$, $\beta = 0$.

$$\left.\begin{aligned} B_t &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \left.\frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}\right|_{\substack{\alpha=1\\ \beta=1}} \\ &= \frac{1}{\tau_0} \delta_{t/\tau_0} \\ &= \delta_t . \end{aligned}\right\} \tag{6.8}$$

$$S(p) = \left.\frac{1}{[1 + (p\tau_0)^{1-\alpha}]^{\beta}}\right|_{\substack{\alpha=1\\ \beta=0}} = 1. \tag{6.9}$$

6. (*iii*) $\alpha = 0,\ \beta = 1.$

$$\left.\begin{aligned} B_t &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}\Bigg|_{\substack{\alpha=0\\ \beta=1}} \\ &= \sum_{n=0}^{\infty} \frac{1}{\tau_0}(-1)^n Pf \frac{(t/\tau_0)_+^n}{\Gamma(n+1)} \\ &= u_0\left(\frac{t}{\tau_0}\right)\frac{1}{\tau_0}\exp\left(-\frac{t}{\tau_0}\right). \end{aligned}\right\} \tag{6.10}$$

$$S(p) = \frac{1}{[1+(p\tau_0)^{1-\alpha}]^\beta}\Bigg|_{\substack{\alpha=0\\ \beta=1}} = \frac{1}{1+p\tau_0}. \tag{6.11}$$

6. (*iv*) $\alpha = 0,\ \beta = 0.$

$$\left.\begin{aligned} B_t &= \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}\Bigg|_{\substack{\alpha=0\\ \beta=0}} \\ &= \frac{1}{\tau_0}\delta_{t/\tau_0} \\ &= \delta_t. \end{aligned}\right\} \tag{6.12}$$

$$S(p) = \frac{1}{[1+(p\tau_0)^{1-\alpha}]^\beta}\Bigg|_{\substack{\alpha=0\\ \beta=0}} = 1. \tag{6.13}$$

The whole situation may be summarised in the following theorem.

Theorem 6.1 *Let* $p \in \boldsymbol{C}^1$, *and* $\alpha, \beta, \tau_0 \in \boldsymbol{R}^1$ *with* $0 \leqq \alpha, \beta \leqq 1$, *and* $0 < \tau_0$. *Let* $\Gamma = \{\sigma \in \boldsymbol{R}^1 \,|\, 0 < \sigma\}$ *and* $\Gamma_1 = \left\{\sigma \in \boldsymbol{R}^1 \,\middle|\, \frac{1}{\tau_0} < \sigma\right\}$. *Let* $B_t \in \mathscr{S}'_t(\Gamma) \cap \mathscr{D}'_R$ *be the unique distribution such that* $\mathscr{L}B_t = S(p)$, *where* $S(p)$ *is the principal branch of the function* $S(p) = \dfrac{1}{[1+(p\tau_0)^{1-\alpha}]^\beta}$.

Then, (*i*) *the distribution* B_t *has its support bounded to the left at* $t = 0$, $\operatorname{supp} B_t \subseteq \{t \in \boldsymbol{R}^1 \,|\, 0 \leqq t\}$,

and (*ii*) *the distribution* B_t *is determined by the infinite series of Riesz distributions*

$$B_t = \sum_{n=0}^{\infty} \frac{1}{\tau_0} \cdot \frac{(-1)^n}{n!} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta)} Pf \frac{(t/\tau_0)_+^{(1-\alpha)(\beta+n)-1}}{\Gamma[(1-\alpha)(\beta+n)]}.$$

In the topology on $\mathscr{S}'_t(\Gamma_1)$ *the series is* (C, 1) *summable on the set* $\{(\alpha,\beta) \in \boldsymbol{R}^2 | 0 \leqq \alpha \leqq 1 \wedge 0 \leqq \beta \leqq 1\}$ *and convergent on the subset* $\{(\alpha,\beta) \in \boldsymbol{R}^2 | 0 \leqq \alpha < 1 \wedge 0 \leqq \beta < 1\}$.

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